MATHEMATICAL METHODS FOR THEORETICAL PHYSICS I HOMEWORK 4

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Problem 1 (Cartesian Tensors in 3-D)**.** .
a)If $\bar{A}$, $\bar{B}$, and $\bar{C}$ are vectors (polar), show that $(\bar{A} \times \bar{B})$ is a psuedovector, and that $\epsilon_{ijk} A_i B_j C_k$ is a psuedoscalar.

Answer. The cross product is a psuedovector because it transforms to maintain its positivity.

$$\begin{aligned} C_i = A_j B_k - A_k B_j \ \ so C_i' &= A_j' B_k' - A_k' B_j' \\ &= (-A_j)(-B_k) - (-A_k)(-B_j) \\ &= A_j B_k - A_k B_j = C_i = C_i' \end{aligned}$$

In Levi-Civita:

$$\begin{aligned} \epsilon_{ijk} A_{jBk} \to \epsilon_{ijk} A_j B_k |a| &\stackrel{?}{=} \epsilon_{ijk}' A_j' B_k' \\ A_j B_k |a| &\stackrel{?}{=} A_j' B_k'^{1} \end{aligned}$$

$|a|$ = the determininant of the transformation matrix, in this case, $a = \mathbf{I}_2$ (Identity matrix, order 2) and so it has a value of 1.

$$\begin{aligned} A_j B_k &\stackrel{?}{=} A_j' B_k' \\ A_j B_k &\stackrel{?}{=} (-A_j)(-B_k) \\ A_j B_k &= A_j B_k. \end{aligned}$$

And so we see that the two quantities are equal, *and* equal to their originiator Now, for the pseudoscalar (which by definition will transorm to its *negative*) we see:

$$\begin{aligned} \epsilon_{ijk} A_i B_j C_k |a| &= \epsilon_{ijk}' A_i' B_j' C_k' \\ &= A_i' B_j' C_k' \\ &= (-A_i)(-B_j)(-C_k) \\ &= -(A_i B_j C_k) = -\epsilon_{ijk} A_i B_j C_k \end{aligned}$$

The transformed scalar is equal to the negative of its originator. Thus, the triple dot product is a pseudoscalar.

b) Let T_{ij} be a second-rank cartesian tensor, and let S_{ij}, A_{ij}, and T be a symmetric (traceless) matrix, an antisymmetric matrix, and a scalar number, respectively. Find expressions for these three entities which can compose T in the following way:

$$T_{ij} = \frac{1}{2} S_{ij} + \frac{1}{2} A_{ij} + \frac{1}{3} \delta_{ij} T.$$

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Answer. This looks suspiciously like Arfken & Weber's equation (3.97) where the matrix is rewritten interms of symmetric and antisymmetric components in the following fashion:

$$\mathbf{A} = \frac{1}{2}[\mathbf{A} + \mathbf{A}^T] + \frac{1}{2}[\mathbf{A} - \mathbf{A}^T].$$

So let's pick these apart to discover if this is in-fact valid...

$$S_{ij} \stackrel{?}{=} [\mathbf{T} + \mathbf{T}^T],\ \mathrm{A}_{ij} \stackrel{?}{=} [\mathbf{T} - \mathbf{T}^T]$$

Suppose so. Suppose that this identities hold true. If this is the case, then we will be able to generate T_{ij} from the addition of $S_{ij} + A_{ij}$:

$$S_{ij} = [\mathbf{T} + \mathbf{T}^T], A_{ij} = [\mathbf{T} - \mathbf{T}^T]$$

$$\begin{aligned} \rightarrow S_{ij} + A_{ij} &= \mathbf{T} + \mathbf{T}^T + \mathbf{T} - \mathbf{T}^T \\ &= T_{ij} + T_{ji} + T_{ij} - T_{ji} = 2T_{ij} \\ \Rightarrow \frac{1}{2}S_{ij} + \frac{1}{2}A_{ij} = \frac{1}{2}[S_{ij} + A_{ij}] &= \frac{1}{2}2T_{ij} = T_{ij} \end{aligned}$$

And thus,

$$S_{ij} = [\mathbf{T} + \mathbf{T}^T], A_{ij} = [\mathbf{T} - \mathbf{T}^T]$$

But There is an additional component added into our original statement. We must account for the $\frac{1}{3}\delta_{ij}T$ term.

Note that the S_{ij} is traceless and so the diagonals sum to zero. T is a pure number, and this is acheivable (in terms of a matrix) only through a determinant or a trace. It seems unlikely that determinant would be involved here, so let's explore the trace. The trace involves elements of the diagonal and thus, may be held to the Kronecker-Delta restriction, (δ_{ij} as indexed according to T_{ij}). We will let the factor of $\frac{1}{3}$ account for the over-summing as a result of the delta. And so with this traceless S_{ij} let us declare

$$T = Tr(T_{ij})$$

As for their transformation properties, let us discuss them in order of rank. The Trace term, $\delta_{ij}T = Tr(T_{ij})$ will transform like a scalar. Psuedoscalars take the form $S' = |a|S$ where $a \equiv \mathbf{I}_1$. In other words, there is a sign change on the psuedoscalar under inversion (but not rotation). Because the trace will not change signs under the inversion, it does not behave like a psuedoscalar, and thus will transform like a scalar. Contrastingly, the A_{ij} term, which can be expressed as $C_i = \frac{1}{2}\epsilon_0 C_{jk}$, is visibly a dual pseudovector, as it will maintain its positive sign through an inversion. As for the S_{ij} term, it is expressed above as an irreducible tensor of Rank 2, which under inversion will carry the negative sign, and thus means that it transforms like a psuedotensor of Rank 2.

c)Show that there is a rotated coordinate system in which S'_{ij} is purely diagonal. What is the value of the trace of S'?

Answer. We know that in order to diagonalize, we can hit a matrix with the following similarity transformation:

$$\mathbf{S}^{\mathbf{Diag}} = \mathbf{P}^T\mathbf{S_{ij}}\mathbf{P}$$

[1] $\epsilon_{ijk} = \epsilon'_{ijk}$ by definition that Levi-Civita is a psuedo tensor, so we can drop this from our expression immediately

So, owning our $\mathbf{S_{ij}}$, we can open it up and ascertain its eigenvalues, use these to find the eigenvectors, and then normalize for the orthonormal eigenvectors, $\mathbf{u_i}$. We also know (from Arfken & Weber 3.3) that we can express any coordinate axis rotation as

$$x'_i = \sum_{j=1}^{3} a_{ij} x_j.$$

Let us now rotate our matrix through two rotation matrices given by $\mathbf{P}$ and $\mathbf{P}^T$. Where (as before)

$$\mathbf{P} \equiv [\mathbf{u}_1, \mathbf{u}_2....\mathbf{u}_n]; \;\; \mathbf{u}_i \equiv \frac{\mathbf{v_i}}{||\mathbf{v_i}||}$$

and $\mathbf{v}_i$ is the eigenvector of corresponding to the ith eigenvalue. Please note that because $\mathbf{P}$ is an orthonormal matrix, its transpose is equal to its inverse.
Accordingly, the trace is given as follows:

$$\begin{aligned} Tr(\mathbf{S^{Diag}}) &= Tr(\mathbf{P}^T\mathbf{S_{ij}}\mathbf{P}) \\ &= Tr(\mathbf{P}\mathbf{P}^T\mathbf{S_{ij}})^2 \\ &= Tr(\mathbf{P}\mathbf{P}^{-1}\mathbf{S_{ij}}) \\ &= Tr(\mathbf{I}\mathbf{S_{ij}}) \\ &= Tr(\mathbf{S_{ij}}) \end{aligned}$$

So we see that the trace of our diagonalized matrix is equal the trace of our original matrix. And in this instance, where S_{ij} is defined as a traceless entity, we see that the trace of our diagonalized matrix is 0.

$$Tr(\mathbf{S_{ij}}) = 0.$$

d) The antisymmetric piece can be written as $A_{ij} = \epsilon_{ijk} v_k$. Show that v_k transforms like a pseudovector.

Answer. Let us write the antisymmetric vector $\mathbf{A}$ as

$$\mathbf{A} \equiv \begin{bmatrix} 0 & A_{12} & A_{31} \\ A_{12} & 0 & A_{23} \\ A_{31} & A_{23} & 0 \end{bmatrix}$$

We know that $\mathbf{A}$ must transform as a vector under rotations from the double contraction of the fifth-rank pseudo tensor $\epsilon_{ijk} C_{mn}$ but that is really a pseudovector from the pseudo nature of ϵ_{ijk}. Specifically the components are given by

$$(C_1, C_2, C_3) = (C_{23}, C_{31}, C_{12})$$

Notice the cyclic order of the indices that comes from the cyclic order of the components of ϵ_{ijk}. This duality, given by the above equation means that our three-dimensional vector product may literally be taken to be either a pseudovector or an antisymmetric second-rank tensor, depending on how we choose to write it out.

We know that A_{ij} is a pseudovector (re:above). So, under transformation we see that

$$-A_{ij} \rightarrow \epsilon_{ijk} v_k > 0$$

ϵijk is a pseudotensor, so this maintains positivity in the inversion. SO, the consequence

[2]Trace arguments are equivalent under cyclic permutation

is that if our entire expression $\epsilon_{ijk}v_k$ must be greater than zero, and if our Levi is to remain positive, then v_k must be positive- i.e. it *must* be a pseudovector.

$$\epsilon ijkv_k \overset{!}{>} 0.$$

Otherwise, suppose not: Suppose A_{ij} is a pseudo matrix but v_k is not a pseudo vector. By assignment, $-A_{ij} = +(\epsilon_{ijk}v'_k)$. Again, ϵ_{ijk} is a pseudotensor, and so remains positive. Now, v_k is supposedly not a pseudo vector, so it will transform negatively:

$$-A_{ij} = -v_k \neg\big(+ (-v_k)\big). \rightarrow\leftarrow$$

Contradiction

Problem 2 (Problems in the delta function)**.** .
a) Evaluate the following:

$$\int_0^\pi dx \int_1^2 dy\delta(\sin x)\delta(x^2 - y^2).$$

Answer. We know that integrals that resemble the following format

$$\int f(x)\delta(x^*)dx = f(a)$$

(where $x^* = g(x)$ such that $g(x) = 0$ at some point $x = a$) can be utilized to simplify integrals. Accordingly, let's make the following substitution:

$$a = x^2, da = 2xdx, dx = \frac{da}{2x} = \frac{da}{2\sqrt{a}}$$

and so...

$$\int_0^\pi dx \int_1^2 dy\delta(\sin x)\delta(x^2 - y^2) = \int_0^\pi \int_1^2 \frac{\delta(\sin\sqrt{a})}{2\sqrt{a}} dy\delta(a - y^2)dady = \int_0^\pi \frac{\delta(\sin y)}{2y} dy$$

Again, let's select $f(y)$ to be $\frac{1}{2y}$, $\delta(siny)$ selects out $y = 0, \pi$. (Both 0 and π are within our bounds, so we may use both– BUT, using 0 will yield an undefined result, so we will use π.)

$$\int_0^\pi \frac{\delta(\sin y)}{2y} dy = \frac{1}{2\pi}$$

b) **Arfken & Weber** *1.15.11 Show that in spherical polar coordinates* $(r, \cos\theta, \phi)$ *the delta function* $\delta(\mathbf{r}_1 - \mathbf{r}_2)$ *becomes*

$$\frac{1}{r_1^2}\delta(r_1 - r_2)\delta(\cos\theta_1 - \cos\theta_2)\delta(\phi_1 - \phi_2).$$

Answer. We know that by definition of the delta that

$$\int\int\int_{AllSpace} \delta(\mathbf{r}_1 - \mathbf{r}_2)r_2^2 dr_2 \sin\theta_2 d\theta_2 d\phi_2 = 1.$$

So, if

$$\delta(r_2 - r_1) = \frac{1}{r_1^2}\delta(r_1 - r_2)\delta(\cos\theta_1 - \cos\theta_2)\delta(\phi_1 - \phi_2)$$

does this hold?

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{r_2^2}\delta(r_2-r_1)\delta(\cos\theta_2-\cos\theta_1)\delta(\phi_2-\phi_1)r_2^2dr_2\sin\theta_2 d\theta_2 d\phi_2 \stackrel{?}{=} 1$$
$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\delta(r_2-r_1)\delta(\cos\theta_2-\cos\theta_1)\delta(\phi_2-\phi_1)dr_2\sin\theta_2 d\theta_2 d\phi_2 \stackrel{?}{=} 1$$

****Note:** Separable integrands, so let's pull them apart:

$$\int_0^{2\pi}\delta(\phi_2-\phi_1)d\phi_2\int_0^{\pi}\delta(\cos\theta_2-\cos\theta_1)\sin\theta_2 d\theta_2\int_0^{R}\delta(r_2-r_1)dr_2 \stackrel{?}{=} 1$$

The first integral selects out $\phi_1 = \phi_2$, the second selects out $\cos\theta_2 = \cos\theta_1$ -or more properly- $\theta_2 = \theta_1$, and the third integral selects out where $r_1 = r_2$. In the case of the ϕ integral, this is true trivially. Similarly, by construction, this is true for the r integral as well. For the θ integral, however, Either the angles are equal on the interval $0 \to \pi$ or on the interval $-\pi \to 0$. In either case, we are okay, because $\cos\theta = -\cos\theta$.
Thus, all 3 of our integrals select out valid quantities, and so all are unity.

$$1 \cdot 1 \cdot 1 \stackrel{\checkmark}{=} 1.$$

Problem 3 (Matrix Problems)**.** .
1) Let U be a unitary matrix with eigenvectors $|x_1>$ and $|x_2>$ belonging to eigenvalues λ_1, λ_2 respectively. Show that:
a) $|\lambda_1| = |\lambda_2| = 1$

Answer. We know that by definition, we have:

$$A|\chi_1>= \lambda_1|\chi_1>$$

But in this discussion we are working with a Unitary matrix. So, more properly:

$$\mathbf{U}|\chi_1>= \lambda_1|\chi_1>$$

Now, let us run through an identity:

$$\begin{aligned}
<\chi_1|\mathbf{U}^\dagger\mathbf{U}|\chi_1> &= <\chi_1|\mathbf{U}^\dagger\lambda_1|\chi_1> \\
<\chi_1|\chi_1> &= <\chi_1|\mathbf{U}^\dagger\lambda_1|\chi_1> \\
<\chi_1|\chi_1> &= \lambda_1^* <\chi_1|\lambda_1|\chi_1> \\
1 &= \lambda_1^*\lambda_1 <\chi_1|\chi_1> \\
&\Rightarrow \lambda_1^*\lambda_1 = 1
\end{aligned}$$

And this is true, because by definition, $||\lambda_1^*\lambda_1|| = 1$.
Now, for the converse. Suppose not. Suppose $_1$ and λ_2 were different.

$$\begin{aligned}
<\chi_2|\mathbf{U}|\chi_1> &= \lambda_1 <\chi_2|\chi_1> \\
<\chi_2|\mathbf{U}^\dagger\mathbf{U}\mathbf{U}|\chi_1> &= \lambda_1 <\chi_2|\chi_1> \\
&= \lambda_2^*\lambda_1 <\chi_2|\mathbf{U}|\chi_1> \\
&\to <\chi_2|\chi_1>
\end{aligned}$$

Which is 0 unless $\chi_2 = \chi_1$, (orthogonality condition), which would equate the eigenvalues, forcing a contradiction. $\to\leftarrow$.

b) If $\lambda_1 \neq \lambda_2, <x_1|x_2>= 0$.

Answer. Proof: If $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2 ... \bar{\mathbf{v}}_r$ are linearlly independent then there exists a least index p such that $\bar{\mathbf{v}}_{p+1}$ is a linear combination of the preceding (linearlly independent) vectors. *And* there exist scalars $c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 + ... + c_p\bar{\mathbf{v}}_p = \bar{\mathbf{v}}_{p+1}$.
Multiplying through by A:

$$c_1 A\bar{\mathbf{v}}_1 + c_2 A\bar{\mathbf{v}}_2 + ... + c_p A\bar{\mathbf{v}}_p = \bar{\mathbf{v}}_{p+1} \tag{1}$$

****Note:** $A\bar{\mathbf{v}}_k = \lambda_l \bar{\mathbf{v}}_k$ for all k.

$$c_1\lambda_1\bar{\mathbf{v}}_1 + c_2\lambda_2\bar{\mathbf{v}}_2 + ... + c_p\lambda_p\bar{\mathbf{v}}_p = \lambda_{p+1}\bar{\mathbf{v}}_{p+1}$$

Multiplying through by λ_{p+1} and subtracting from our equation (1):

$$c_1(\lambda_1 - \lambda_{p+1})\bar{\mathbf{v}}_1 + c_2(\lambda_2 - \lambda_{p+1})\bar{\mathbf{v}}_2 + ... + c_p(\lambda_p - \lambda_{p+1})\bar{\mathbf{v}}_p = 0$$

Since $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2 ... \bar{\mathbf{v}}_r$ are linearlly independent, c_i's are all 0. But none of the factors $\lambda_1 ... \lambda_{p+1}$ are 0 because they are all distinct. Hence, $c_i = 0$ for all $i = 1...p$. *But* then our equation (1) reads that $V_{p+1} = 0$. *Contradiction.* $\rightarrow\leftarrow$. Thus, $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2 ... \bar{\mathbf{v}}_r$ must be linearly dependent and can not be linearly independent.

2) Let A be a 3x3 matrix

$$\mathbf{A} \equiv \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

a) Find the eigenvalues and normalized eigenvectors of A

Answer.

$$\mathbf{A} \equiv \begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix} = 0$$

Generates the following characteristic equation

$$(2-\lambda)[(2-\lambda)(2-\lambda) - 1] + (-1)(2-\lambda) = 0$$
$$(2-\lambda)[(2-\lambda)^2 - 2] = 0$$

So, λ_1 =2, or

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = 2 \pm \sqrt{2} = \lambda_{2,3}.$$

Accordingly, the proper eigenvectors are as follows:

$$\lambda_1 = 2, \bar{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \lambda_2 = 2 + \sqrt{2}, \bar{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \lambda_3 = 2 - \sqrt{2}, \bar{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

The orthonormal basis vectors are as follows:

$$\bar{\mathbf{u}}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \bar{\mathbf{u}}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \bar{\mathbf{u}}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

b) Construct an orthogonal matrix O which diagonalizes **A**

Answer.

$$\mathbf{P} = [\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \ \mathbf{D} \stackrel{!}{=} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & & \\ & 2+\sqrt{2} & \\ & & 2-\sqrt{2} \end{bmatrix}$$

$\mathbf{P}$ is square, has orthonormal columns, so $\mathbf{P}$ is orthogonal, $\mathbf{P}^{-1} = \mathbf{P}^T$.

$$\mathbf{P}^T\mathbf{AP} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{2}} & 1+\frac{\sqrt{2}}{2} & 1-\frac{\sqrt{2}}{2} \\ 0 & 1-\sqrt{2} & -1+\sqrt{2} \\ -\frac{2}{\sqrt{2}} & 1+\frac{\sqrt{2}}{2} & 1-\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{2}+\frac{2}{2} & 0 & 0 \\ 0 & (1+\frac{\sqrt{2}}{2})2 & 0 \\ 0 & 0 & (1-\frac{\sqrt{2}}{2})2 \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2+2\sqrt{2} & 0 \\ 0 & 0 & 2-\sqrt{2} \end{bmatrix}$$

c) Use spectral decomposition to
-show that $\sum_{i=1}^{3} |e_i >< e_i| = \mathbf{I}$, *and* $\sum_{i=1}^{3} \lambda_i |e_i >< e_i| = A$

Answer. Suppose $\mathbf{A} = \mathbf{PAP}^{-1}$ where the columns of $\mathbf{P}$ are the orthonormal eigenvectors $\bar{\mathbf{u}}_1, ..., \bar{\mathbf{u}}_n$, and the corresponding eigenvalues $\lambda_1, ..., \lambda_n$ are in the diagonal matrix $\mathbf{D}$. Then, since $\mathbf{P}^{-1} = \mathbf{P}^T$:

$$\mathbf{A} = \mathbf{PAP}^T = \begin{bmatrix} \bar{\mathbf{u}}_1 & .... & \bar{\mathbf{u}}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}}_1^T \\ \vdots \\ \bar{\mathbf{u}}_n^T \end{bmatrix}$$

$$= \lambda_1 \bar{\mathbf{u}}_1 \bar{\mathbf{u}}_1^T + \lambda_2 \bar{\mathbf{u}}_2 \bar{\mathbf{u}}_2^T + ... + \lambda_n \bar{\mathbf{u}}_n \bar{\mathbf{u}}_n^T.$$

$\mathbf{D}$ is a square (diagonal) matrix, so we can do it out:

$$\mathbf{AD}^{-1} = \mathbf{PP}^T = \bar{\mathbf{u}}_1 \bar{\mathbf{u}}_1^T + \bar{\mathbf{u}}_2 \bar{\mathbf{u}}_2^T + ... + \bar{\mathbf{u}}_n \bar{\mathbf{u}}_n^T.$$

In our matrix notation:

$$\mathbf{PP}^T = \sum_{i=1}^{3} \lambda_i |e_i >< e_i| \frac{1}{\lambda_i} = \sum_{i=1}^{3} |e_i >< e_i|$$

Note:

$$\begin{aligned} \bar{\mathbf{u}}_1 \bar{\mathbf{u}}_1^T \to \mathbf{PP}^T = \mathbf{PP}^{-1} &= \bar{\mathbf{u}}_1 \bar{\mathbf{u}}_1^{-1} + \bar{\mathbf{u}}_2 \bar{\mathbf{u}}_2^{-1} + ... + \bar{\mathbf{u}}_n \bar{\mathbf{u}}_n^{-1} \\ &= \mathbf{1}_1 + \mathbf{1}_2 + ... + \mathbf{1}_n \\ &= \mathbf{I}_n \stackrel{\checkmark}{=} \sum_{i=1}^{3} |e_i >< e_i| \end{aligned}$$

Now calculate $\mathbf{A}^{-1}$ *and exp*$\mathbf{A}$.

Answer.

$$\begin{aligned} \mathbf{A} &= \lambda_1 \bar{\mathbf{u}}_1 \bar{\mathbf{u}}_1^T + ... + \lambda_n \bar{\mathbf{u}}_n \bar{\mathbf{u}}_n^T \\ \mathbf{A} &= \mathbf{P}\mathbf{D}\mathbf{P}^T \\ \mathbf{A}^{-1} &= (\mathbf{P}^T)^{-1}\mathbf{D}^{-1}\mathbf{P}^{-1\,3} \\ &= (\mathbf{P}^{-1})^T\mathbf{D}^{-1}\mathbf{P}^{-1} \\ &= (\mathbf{P}^T)^T\mathbf{D}^{-1}\mathbf{P}^T \\ &= \mathbf{P}\mathbf{D}^{-1}\mathbf{P}^T \end{aligned}$$

Note:

$$\big(diag(a_1, ..., a_n)\big)^{-1} = diag(\frac{1}{a_1}, ..., \frac{1}{a_n})$$

$\mathbf{A}^{-1} = \mathbf{P}\mathbf{D}^{-1}\mathbf{P}^T = \mathbf{P}\mathbf{D}^{-1}\mathbf{P}^{-1}$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{a} & 0 \\ 0 & 0 & \frac{1}{b} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix}$$

Where

$$a \equiv 2 + \sqrt{2}, \;\; b \equiv 2 - \sqrt{2}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{2}} & 0 & \frac{2}{\sqrt{2}} \\ \frac{1}{2a} & -\frac{\sqrt{2}}{2a} & \frac{1}{2a} \\ \frac{1}{2b} & \frac{\sqrt{2}}{2b} & \frac{1}{2b} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \frac{1}{4a} + \frac{1}{4b} & -\frac{\sqrt{2}}{4}(\frac{1}{a} + \frac{1}{b}) & 1 + \frac{1}{4a} + \frac{1}{4b} \\ -\frac{\sqrt{2}}{4}(\frac{1}{a} - \frac{1}{b}) & \frac{1}{2a} + \frac{1}{2b} & +\frac{\sqrt{2}}{4}(\frac{1}{a} + \frac{1}{b}) \\ -\frac{1}{\sqrt{2}} + \frac{1}{4a} + \frac{1}{4b} & -\frac{\sqrt{2}}{4}(\frac{1}{a} - \frac{1}{b}) & -\frac{1}{\sqrt{2}} + \frac{1}{4a} + \frac{1}{4b} \end{bmatrix} = \mathbf{A}^{-1}$$

3) Show that if A is a Hermitian martix, then $det[exp(A)] = exp[tr(A)]$.

Answer. The diagonalization theorem says that a similarity transformation

$$\mathbf{A}' = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$$

Where $Tr(\mathbf{A}') = Tr(\mathbf{A})$ (as shown in Problem 1.3).

$$Tr(\mathbf{A}') = A'_1 + A'_2 + ... + A'_n$$

This is the sum of all eigenvalues of $\mathbf{A}'$. Whereas the *determinant* is the *product* of the eigenvalues:

$$det(\mathbf{A}) = A'_1 \cdot A'_2 \cdot ... \cdot A'_n$$

Let us define:

$$expA = 1 + \frac{1}{1!}A + \frac{1}{2!}A^2 + ... = n \overset{\lim}{\rightarrow} \infty \; (1 + \frac{n}{N})^N$$

Now, the eigenvalues of $exp(A) = exp(A')$. So:

$$det(expA) = \prod_i e^{A_i} = exp(\sum_i A_i) = exp(Tr(\mathbf{A})) \overset{\checkmark}{=} det(expA).^4$$

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[3] $(\mathrm{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Problem 4 (Infinite Sums)**.** .
(a)Show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$$

Hint: *Show by mathematical induction that* $s_m = \frac{m}{(2m+1)}$.

Answer. 1) Show that $s_m = \frac{m}{(2m+1)}$:
i) Show P(1):

$$\frac{1}{(2-1)(2+1)} = \frac{1}{3} = \frac{1}{2(1)+1}$$

ii) Presume P(m) to be true. That is, presume $s_m = \frac{m}{(2m+1)}$.
iii) Show P(m+1) to be true.

$$\sum_{m=1}^{\infty} \frac{1}{(2m+1)(2m-1)} + \frac{1}{(2(m+1)-1)(2(m+1)-1)} \stackrel{?}{=} \frac{m+1}{(2(m+1)+1)}$$

By presumption:

$$\sum_{m=1}^{\infty} \frac{1}{(2m+1)(2m-1)} = \frac{m}{(2m+1)},$$

So our equivalence becomes

$$\begin{aligned}
\frac{m}{(2m+1)} + \frac{1}{(2m+1)(2m+3)} &\stackrel{?}{=} \frac{m+1}{(2m+3)} \\
\frac{m(2m+3)}{(2m+1)} + \frac{(2m+3)}{(2m+1)(2m+3)} &\stackrel{?}{=} \frac{(2m+3)m+1}{(2m+3)} \\
\frac{m(2m+3)}{(2m+1)} + \frac{1}{(2m+1)} &\stackrel{?}{=} (m+1) \\
m(2m+3)+1 &\stackrel{?}{=} (2m+1)(m+1) \\
2m^2+3m+1 &\stackrel{\checkmark}{=} 2m^2+2m+1m+1
\end{aligned}$$

And now that we have shown that this is a reasonable substitution, let us show that s_m has a limit of $\frac{1}{2}$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{m}{2m+1} &= \lim_{n\to\infty} \frac{\frac{d}{dm}2m}{\frac{d}{dm}m + \frac{d}{dm}1}^{5} \\
&= \lim_{n\to\infty} \frac{1}{2+0} \stackrel{\checkmark}{=} \frac{1}{2}.
\end{aligned}$$

(b)Show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

[4]$(e^a e^b = e^{a+b})$

[5]By L'Hopital's rule, $\lim_{n\to\infty} \frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{\frac{d}{dx}f(x)}{\frac{d}{dx}g(x)}$.

Find the partial sum s_m and verify its correctness by mathematical induction.

Answer. 1) Find s_m. Let's do this by expanding out our first few terms:

$$\frac{1}{1(1+1)}+\frac{1}{2(2+1)}+\frac{1}{3(3+1)}+\frac{1}{4(4+1)}=\frac{1}{12}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}$$
$$s_1=\frac{1}{2}, s_2=\frac{1}{2}+\frac{1}{6}, s_3=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}, s_4=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$$
$$s_1=\frac{1}{2}, s_2=\frac{4}{6}, s_3=\frac{9}{12}, s_4=\frac{16}{20}$$
$$\Rightarrow s_m=\frac{n^2}{n(n+1)}$$

Now to verify that this is the equivalence of our given sum:
i) Show P(1):

$$\frac{1}{1(1+1)}=\frac{1}{2}=\frac{1^2}{1(1+1)}$$

ii) Presume P(m) to be true. That is, presume $s_m=\frac{m^2}{m(m+1)}$.
iii) Show P(m+1) to be true.

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\frac{1}{(n+1)(((n+1)+1)}\stackrel{?}{=}\frac{(n+1)^2}{(n+1)((n+1)+1)}$$
$$\frac{n^2}{n(n+1)}+\frac{1}{(n+1)(((n+1)+1)}\stackrel{?}{=}\frac{(n+1)^2}{(n+1)((n+1)+1)}$$
$$\frac{n^2}{n(n+1)}+\frac{1}{(n+1)(n+2)}\stackrel{?}{=}\frac{(n+1)^2}{(n+1)(n+2)}$$
$$\frac{(n+2)n^2}{n(n+1)}+\frac{1}{(n+1)}\stackrel{?}{=}\frac{(n+1)^2}{(n+1)}$$
$$\frac{(n+2)n}{(n+1)}+\frac{1}{(n+1)}\stackrel{?}{=}(n+1)$$
$$(n+2)n+1\stackrel{?}{=}(n+1)^2$$
$$(n^2+2n)+1\stackrel{\checkmark}{=}(n^2+2n+1)$$

Again, to check the limit:

$$\lim_{n\to\infty}\frac{m^2}{m(m+1)}=\lim_{n\to\infty}\frac{\frac{d}{dm}m^2}{\frac{d}{dm}m^2+\frac{d}{dm}m}$$
$$\lim_{n\to\infty}\frac{2m}{2m+1}=\lim_{n\to\infty}\frac{\frac{d}{dm}2m}{\frac{d}{dm}2m+\frac{d}{dm}1}$$
$$\lim_{n\to\infty}\frac{2}{2}=1$$