\[ \sum_{i,p} \delta q_i \left[ \frac{d}{dt} \left( m_i v_i \frac{\partial \mathcal{L}}{\partial \dot{v}_i} \right) - m_i v_i \frac{\partial \mathcal{L}}{\partial q_i} \right] \]

\[ = \sum_{i,p} \delta q_i \left[ \frac{d}{dt} \left( m_i v_i^2 \frac{\partial}{\partial v_i} \right) - \frac{\partial}{\partial q_i} v_i^2 \right] \]

\[ = \sum_{i,p} \delta q_i \left[ \frac{d}{dt} \left( \frac{\partial}{\partial \dot{q}_i} \right) - \frac{\partial}{\partial q_i} \right] \]

\[ = \sum_{i,p} \delta q_i \left[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} \right] \]

\[ = 0 \]

\[ \delta q_i \text{ independent} \]

\[ \delta q_i \text{ independent} \]

\[ = 0 \]

\[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} - Q_i = 0 \]

\[ \text{for } i = 1, \ldots, 3N \text{ if } \frac{\partial \mathcal{L}}{\partial q_i} = 0 \]

- Conservative forces:

\[ Q_i = -\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \]

\[ \left( \sum_i \mathcal{F}_i \cdot \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = -\sum_i \mathcal{F}_i \cdot \frac{\partial \mathcal{L}}{\partial q_i} \right) \]

\[ \text{for } \mathcal{V} \text{ independent of } q_i \frac{\partial \mathcal{V}}{\partial q_i} = 0 \]

\[ = \mathcal{V} \text{ Lagrangian} \]

\[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \]

\[ \text{Lagrangian Equations} \]
Example:

One dof: \( x \)

\[
T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 \quad \text{with } y = l \sin \alpha
\]

\[
V = m g y = m g l \sin \alpha
\]

\[
\frac{dL}{dx} = m l \ddot{\alpha} \quad \frac{dL}{d\alpha} = -m g l \cos \alpha
\]

\[
\frac{d}{dt} \left( \frac{dL}{dx} \right) - \frac{dL}{d\alpha} = m l (l \ddot{\alpha} + g \cos \alpha) = 0
\]

\[
\ddot{\alpha} = -\frac{g}{l} \cos \alpha
\]

Final case (also non-holonomic)

(Refer ...)

\[
\]
(c) Hamilton's Principle

Def: "Configuration space": also space spanned by all $q_i$'s ($\mathbb{R}^D$), "motion" happens in configuration space.

Hamilton's principle:

The action

$$ S = \int_{t_1}^{t_2} L \, dt $$

is stationary (i.e. min) for the path of motion.

$$ \Rightarrow \delta S = \delta \int_{t_1}^{t_2} L (q_1, ..., q_i, ..., t) \, dt = 0 $$

"variation" $\delta$
leads to Lagrange's Eqs! (instead of Newton's eqs)

Advantages (of this approach):
this is independent of choice of variables $q_i$!

mathematical tools for calculus of variational calculus: Goldstein 2.2.
Derivation of Laplace's Eq's via the variational principle:

\[ \delta \int_{t_1}^{t_2} L dt = 0 \quad \text{with} \quad \delta t = 0 \quad \text{(no variation of time)} \]
\[ \delta q_i \bigg|_{t_1} = \delta q_i \bigg|_{t_2} = 0 \quad \text{(path is fixed at end points)} \]

\[ \delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \left( \sum_{\alpha} \frac{\partial L}{\partial q_\alpha} \delta q_\alpha + \sum_{\alpha} \frac{\partial L}{\partial \dot{q}_\alpha} \delta \dot{q}_\alpha \right) dt \]

with
\[ \frac{d}{dt} \delta q_\alpha = \delta \frac{d}{dt} q_\alpha = \delta \dot{q}_\alpha \]

then:
\[ \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_\alpha} \delta \dot{q}_\alpha dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial q_\alpha} \delta q_\alpha dt - \int_{t_1}^{t_2} \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} \right) \delta \dot{q}_\alpha dt = 0 \]

\[ \delta S = \int_{t_1}^{t_2} \sum_{\alpha} \left( \frac{\partial L}{\partial q_\alpha} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} \right) \delta q_\alpha dt = 0 \]
For $q$ independent $\Rightarrow \delta q$ independent

$\Rightarrow \sum_\alpha \left[ ... \right. \text{vanishes only if each coeff. of } \delta q \text{ vanishes} \Rightarrow$

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_\alpha} \right) - \frac{\partial L}{\partial q_\alpha} = 0 \quad \forall \alpha
\]

D) Lagrange Eq's with Constraints

(2) holonomic

\[ f_k (q_1, \ldots, q_n, t) = 0 \quad k = 1, \ldots, s \]

Introduce: $\lambda_k$: "Lagrange multipliers"

\[ S' = \int_{t_1}^{t_2} \left( L + \sum_{k=1}^{s} \lambda_k f_k \right) dt \]

1) variation by $\lambda_k$:

\[ \delta S' \bigg|_{q=\text{const}} = \int_{t_1}^{t_2} \left( \sum_k \left( \frac{\partial L}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial \dot{q}_\alpha} + \sum_k \lambda_k \frac{\partial f_k}{\partial q_\alpha} \right) \right) \delta q_\alpha = 0 \]

If independent $\lambda = \Rightarrow f_k = 0 \quad \forall \alpha$

2) variation by $q_\alpha$:

\[ \delta S' \bigg|_{\text{const}} = \int_{t_1}^{t_2} \left( \sum_{\alpha} \left( \frac{\partial L}{\partial q_\alpha} \frac{\partial q_\alpha}{\partial \dot{q}_k} + \sum_k \lambda_k \frac{\partial f_k}{\partial q_\alpha} \right) \right) \delta q_\alpha \]