and $H = eq$ can be equally split.

Def: $H = eq$ is "completely separable" if all coordinates are separable.

Example: 2D harmonic oscillator

Example: Kepler's problem

1) Show that every cyclic variable is separable:

$q_i$ cyclic $\Rightarrow p_i = \text{constant} = \phi$

$\Rightarrow H(q_2 - q_m, \frac{\partial W}{\partial q_2} - \frac{\partial W}{\partial q_m}) = \chi_i \quad \Box$

Try $W = W_i(q_i, \chi) + W(q_2 - q_m, \chi)$

$\Rightarrow \frac{\partial W}{\partial q_i} = p_i = \phi \Rightarrow W_i = \phi q_i$

$W = W_i + \phi q_i$, and $\chi$ depends only on $W_i$

2) General conditions for separability used for separability in orthogonal coordinate systems
1) Hamiltonian is conserved.

ii) Lagrangian is no more than quadratic in generalized velocities, thus

\[ H = \frac{1}{2} (\ddot{\mathbf{p}} - \dot{\mathbf{q}}) \mathbf{T}^{-1} (\ddot{\mathbf{p}} - \dot{\mathbf{q}}) + V(\mathbf{q}) \]

iii) \( a : = a_i = a_i(q_i) \) (only function of corresponding coord.)

iv) \( V(\mathbf{q}) = \sum_i \frac{V_i(q_i)}{T_{ii}} \)

v) Function \( \phi \) with \( \sum_i \phi_i \frac{\partial W_j}{\partial q_i} = \frac{1}{T_{ii}} \)

where \( \frac{\partial W_i}{\partial q_i} - a_i = 2 \sum_k T_{ik} \phi_{ij} \phi_{kj} \)

with \( r_i = \text{const} \), and both \( \phi_{ii} \) and \( \phi_{ij} \) depend only on \( q_i \).

"Stäckel conditions"

If all Stäckel conditions are satisfied, characteristic function is separable: \( W = \sum_i W_i(q_i) \)

\[
\left( \frac{\partial W_i}{\partial q_i} - a_i \right)^2 = -2V_i(q_i) + 2\phi_{ij} \phi_{ji}
\]
3. Application de Kepler's problem

\[ H = \frac{1}{2m} \left( p_r^2 + \frac{p_{\phi}^2}{r^2} \right) + V(r) \]

is cyclic in \( \phi \). \( \Rightarrow \)

\[ W = W_r(r) + \alpha_4 \phi \quad \text{(where } \alpha_4 = p_{\phi} \text{)} \]

\[ = \int \left( \frac{dW_r}{dr} \right)^2 + \frac{\alpha_4^2}{r^2} + 2mV(r) \right) = 2m \int E \]

\[ = \int \frac{dW_r}{dr} = \sqrt{2m(E-V)} - \frac{\alpha_4^2}{r^2} \quad \text{and thus} \]

\[ W_r = \int dr \sqrt{2m(E-V)} - \frac{\alpha_4^2}{r^2} + \alpha_4 \phi \]

Then (as for harmonic oscillator example)

\[ \left[ t + \beta_1 = \frac{\partial W}{\partial E} = \int dr \sqrt{2m(E-V)} - \frac{\alpha_4^2}{r^2} \right] \quad \text{and} \]

\[ \beta_2 = \frac{\partial W}{\partial \alpha_4} = -\int dr \left( \frac{\alpha_4}{r^2} \sqrt{2m(E-V)} - \frac{\alpha_4}{r^2} \right) + \phi \]

These give (for \( n = \frac{1}{r} \)) the results needed and etc. etc. previously obtained, with \( \phi \rightarrow \phi ; \beta_2 \rightarrow 2\phi \).
4) Same problem, but without prior 2D knowledge:

\[ H = \frac{1}{2m} \left( \frac{p_r^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \theta} \right) + V(r) \]

If separable (test) then \( W = W_r + W_\theta + W_\phi \)

\[ \phi \text{ cyclic } \Rightarrow W_\phi = \alpha \phi \phi' = 0 \]

\[ \left( \frac{\partial W_r}{\partial r} \right)^2 + \frac{1}{r^2} \left( \left( \frac{\partial W_r}{\partial \theta} \right)^2 + \frac{\alpha \phi^2}{\sin^2 \theta} \right) + 2mV = 2mE \]

only dependence

\[ = \text{ const} \]

\[ = \alpha \left( \frac{\partial W_r}{\partial \theta} \right)^2 + \frac{\alpha \phi^2}{\sin^2 \theta} = \alpha \phi^2 \]

and \( \left( \frac{\partial W_r}{\partial r} \right)^2 + \frac{\alpha \phi^2}{r^2} + 2mV = 2mE \)

(same as before)

5) Stereoball coord's for spherical coord's:

\[ T_r = m; \quad T_{\theta \theta} = -mr^2; \quad T_{\phi \phi} = m r^2 \sin^2 \theta \]

\[ \Rightarrow V(r) = V_r(r) + \frac{V_\theta(\theta)}{r^2} + \frac{V_\phi(\phi)}{r^2 \sin^2 \theta} \]
• Action-angle variables

Consider special type of problems:

periodic motion, where mostly
frequency is of interest.

Introduce: \( J_1(x, \ldots, x_n) \) "action variables".

Assume conservative system (here only one coord.)

\[ H(q, p) = \omega, \quad \Rightarrow \quad p = p(q, x) \quad \Rightarrow \]

What is "periodic motion"?

1.) closed orbits ⇒ \( q, p \) are periodic functions of time with equal period. ⇒ "libration" (ex: har.m.ox)

2.) \( p \) is periodic function of \( q \) with period \( T \). (Values of \( q \) are not bounded)
   ⇒ "rotation". (ex: \( q \) is phase angle)

Sometimes: 1.) and 2.) at same time, e.g. \( \theta \) pendulum

For both, 1.) and 2.), introduce
"action variable" \( I = \int p dq \) (over one period)
with \( p = p(q, \alpha) \Rightarrow \alpha \cdot = H = H(q) \Rightarrow \)
\( W = W(q, \alpha) \)
unnormalized canonical coord. to \( \alpha \):
\[
W = \frac{\partial W}{\partial \alpha} \quad \Rightarrow \quad W = \frac{\partial N}{\partial \alpha} = v
\]
and \( v \) is constant, let of \( f \) only!
\( \Rightarrow \quad v = vt + \delta \).
So \( v \), \( f \) and \( w \) special cases of \( P_i, Q_i \) only...

What is physical meaning of \( v \)?

**def:** \( \Delta W = \int \frac{\partial W}{\partial \alpha} \, dq \) one period

\[
= \int \frac{\partial W}{\partial f} \, dq
\]

But \( f \) is constant of \( q = 0 \)

\[
\Delta W = \frac{df}{df} \int \frac{\partial W}{\partial q} \, dq = \frac{df}{df} \int \frac{\partial W}{\partial q} \cdot dq = 1
\]

\( \Rightarrow \quad \Delta W = v \Delta t \quad \Rightarrow \quad \frac{1}{v} = \text{period} \quad \Rightarrow \quad \omega = \text{frequency} !
This is found without complete solution - calc!

Example: hcm. arc:

\[ J = \int p d\theta = \int \sqrt{2mE - k^2 \theta^2} \, d\theta \]

Subs h in integration \( \theta = \sqrt{\frac{2E}{k}} \sin \alpha \)

\[ J = \frac{KE}{\sqrt{k/m}} \int_0^{2\pi} \cos^2 \alpha \, d\alpha = \frac{2\pi E}{\sqrt{k/m}} \]

\[ E = \frac{\sqrt{k/m} J}{2\pi} \, \text{curd} \]

\[ \nu = \frac{2H}{\partial J} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \]