VII Hamilton–Jacobi Theory

Find $q = q(q_0, p_0, t)$ \(\text{as canonical transformation} \quad p = p(q_0, p_0, t)\)

- Hamilton–Jacobi Equations

How to find such a transformation?

Assume simplest case:

\[ K = 0 \implies Q_i = \frac{\partial K}{\partial J_i} = 0 \]
\[ \dot{Q}_i = -\frac{\partial K}{\partial q_i} = 0 \]
\[ K = H + \frac{\partial F}{\partial t} = 0 \]

Assume $F = F_2(q, p, t)$ (which contains also identity – drags)

\[ 0 = \dot{P}_i = \frac{\partial F_2}{\partial q_i} , \text{ thus} \]

\[ H + \frac{\partial F}{\partial t} = \left[ H \left( \{ q_i \} , \{ \frac{\partial F_2}{\partial q_i} \} , t \right) + \frac{\partial F_2}{\partial t} = 0 \right. \]

Partial differential eq. in $q, t$
These are \( n+1 \) variables = \( \delta \) generating function in this context is called "Hamilton's principal function" and denoted \((F_2 \rightarrow S)\) "\( S \)".

\( S \) should contain \( n+1 \) integration constants. But: \( S \) does not really make, only the partial derivatives.

With \( S \) now determined only up to (unimportant) constant \( \Rightarrow \) only \( n \) independent integration constants \( \alpha_1, \ldots, \alpha_n \) remain.

\[
\Rightarrow S = S(q_1, \ldots, q_n; \alpha; \ldots; \alpha_n; t)
\]

Hamilton's principal function

We want \( P_i = \text{constant} \Rightarrow \)

choose \( P_i = \alpha_i \) (or, similarly, \( P_i = \delta_i (\alpha_1, \ldots, \alpha_n) \)) - this does not make fundamental difference in further derivation, thus choose \( P_i = \alpha_i \)
\[ p_i = \frac{\partial S}{\partial q_i}, \quad Q_i = \frac{\partial S}{\partial x_i} = \beta_i = \text{const} \]
solve these \( \dot{q}_i = q_i (x, \beta; t) \)
\( p_i = p_i (x, \beta; t) \)

**Addendum:** \( S = \int dt + \text{constant} \)

**Proof:** \[
\frac{ds}{dt} = \frac{\partial S}{\partial q_i} \dot{q}_i + \frac{\partial S}{\partial t} = p_i \dot{q}_i - H = L \quad \Box
\]

Note that this cannot be used to shorten the derivation; however, since for the integration over \( L(q, q, t) \) the explicit form of \( q \) is needed.

**Example: Harmonic Oscillator**

\[
H = \frac{1}{2m} (p^2 + m^2 \omega^2 q^2) = E
\]

\[
H_j: \quad \frac{1}{2m} \left[ (\ddot{q})^2 + m^2 \omega^2 q^2 \right] + \frac{\partial S}{\partial t} = 0
\]

\( \text{not explicitly time-dependent} \Rightarrow \frac{\partial S}{\partial t} \text{ not expl. dep.} \)

\( \Rightarrow \frac{\partial S}{\partial t} = -\alpha \)
\[ S(q,p,t) = W(q,p) - \alpha t \]

[Note: this is always possible for Hamiltonians that are not explicitly time-dependent. \( W \) is called a "characteristic function."]

\[ \frac{1}{2m} \left[ \left( \frac{\partial W}{\partial q} \right)^2 + m \omega^2 q^2 \right] = \alpha (E!) \]

\[ W = \sqrt{2m} \int dq \sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}} \quad \text{and} \]
\[ S = \sqrt{2m} \int dq \sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}} - \alpha t \]

\[ \beta' = \frac{2S}{\beta} = \sqrt{\frac{m\omega}{2\alpha}} \int dq \sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}} - t \]

\[ \beta' + t = \frac{1}{\omega} \arcsin \left[ \sqrt{\frac{m\omega}{2\alpha}} q \right] \]

Define \( \beta' = \beta'w \Rightarrow \]

\[ q = \sqrt{\frac{2\omega}{m\omega^2}} \sin (wt + \phi) \]

\[ \frac{p}{q} = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{v_{\text{max}} - \omega^2 q^2} = \sqrt{v_{\text{max}}^2 \cos^2 (wt + \phi)} \]

\[ P = \alpha = \text{Energy} ; \quad \phi = \frac{\phi_0}{\omega} = \text{phase offset} \]
Determine \( q, p \) as functions of \( q_0, p_0 \):
\[
2m x = p_0^2 + m^2 \omega^2 q_0^2
\]
plug into
\[
\frac{q_0}{p_0} = \frac{1}{m \omega} \tan \beta \int 0, 0 \checkmark
\]

\( \Rightarrow \) Hamilton's principal function \( S \) is generator here of canonical transformation to new coordinates that measure phase offset and a canonical momentum that measures total energy.

- **Separation of variables**

What did we gain?

\( \Rightarrow \) If Eq. is separable by variables:

**Def:** A coordinate \( q_i \) is *separable* if

\( S \) can be split into two additive parts, one of which depends only on \( q_i \), the other is independent of \( q_i \) (for \( i = 1 \):

\[
S = S_i (q_1, x_1, t) + S' (q_2...q_n, x, t)
\]
and $H_f - c_q$ can be equally split.

**Def:** $H_f - c_q$ is "completely separable" if all coordinates are separable.

**Example:** 2D harmonic oscillator (see 10.2 - second part)