10

Assignment 4 - Solution

a)

There are 2 variable transformations:

$$x_i \to y_i = x_i - (A^{-1})_{ij} J_j \quad \text{where} \quad \frac{\partial y_i}{\partial x_j} = \delta_{ij} = \frac{\partial x_i}{\partial y_j}$$

$$\Rightarrow \left|\frac{\partial(x_1 \dots x_n)}{\partial(y_1 \dots y_n)}\right| = \det\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n}\end{pmatrix} = 1$$

and $y_i \to z_i = (O^{-1})_{ik} y_k$, where $\frac{\partial y_i}{\partial z_j} = O_{ij}$

$$\Rightarrow \left|\frac{\partial(y_1 \dots y_n)}{\partial(z_1 \dots z_n)}\right| = \det O$$

Now, check the exponent:

$$-\frac{1}{2} x_i A_{ij} x_j + x_i J_i = -\frac{1}{2}\left(y_i + (A^{-1})_{ik} J_k\right) A_{ij} \left(y_j + (A^{-1})_{jl} J_l\right)$$

$$+ \left(y_i + (A^{-1})_{im} J_m\right) J_i = \quad \left((A^{-1})_{ik} = (A^{-1})_{ki}\right)$$

$$= -\frac{1}{2}\Big[y_i A_{ij} y_j + J_k \underbrace{(A^{-1})_{ki} A_{ij}}_{\delta_{kj}} y_j + y_i \underbrace{A_{ij} (A^{-1})_{jl}}_{\delta_{il}} J_l$$

$$+ J_k (A^{-1})_{ki} \underbrace{A_{ij} (A^{-1})_{jl}}_{\delta_{il}} J_l \Big] + y_i J_i + J_i (A^{-1})_{im} J_m$$

$$= -\frac{1}{2}\left[y_i A_{ij} y_j + \cancel{y_j J_j} + \cancel{y_i J_i} + J_i (A^{-1})_{ij} J_j\right] + \cancel{y_i J_i} + J_i (A^{-1})_{ij} J_j$$

$$= -\frac{1}{2} y_i A_{ij} y_j + \frac{1}{2} J_i (A^{-1})_{ij} J_j$$

and

$$\frac{1}{2} y_i A_{ij} y_j = \frac{1}{2} O_{ik} z_k A_{ij} O_{jl} z_l$$

$$= \frac{1}{2} z_k \underbrace{(O^T)_{ki} A_{ij} O_{jl}}_{= a_k \delta_{kl}} z_l = \frac{1}{2} \sum_i a_i z_i^2$$

where O is chosen to be orthogonal $O^T = O^{-1}$, and to diagonalize A,

$$O^T A O = \tilde{A} = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}, \quad \det O = 1$$

$$\Rightarrow \int \frac{dx_1 \dots dx_n}{\sqrt{2\pi}^n} e^{-\frac{1}{2} x_i A_{ij} x_j + x_i J_i} = e^{\frac{1}{2} J_i (A^{-1})_{ij} J_j} \left| \frac{\partial(x_1 \dots x_n)}{\partial(y_1 \dots y_n)} \right|$$

$$\cdot \left| \frac{\partial(y_1 \dots y_n)}{\partial(z_1 \dots z_n)} \right| \cdot \int \frac{dz_1 \dots dz_n}{\sqrt{2\pi}^n} e^{-\frac{1}{2} \sum_i a_i z_i^2}$$

$$= e^{\frac{1}{2} J_i (A^{-1})_{ij} J_j} \cdot \prod_{i=1}^n \left(\int \frac{dz_i}{\sqrt{2\pi}} e^{-\frac{a_i}{2} z_i^2} \right) = e^{\frac{1}{2} J_i (A^{-1})_{ij} J_j} \prod_{i=1}^n \frac{1}{\sqrt{a_i}}$$

$$= \frac{e^{\frac{1}{2} J_i (A^{-1})_{ij} J_j}}{\sqrt{\det A}},$$

where $\prod \frac{1}{\sqrt{a_i}} = \left(\det \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}\right)^{-1/2} = (\det A)^{-1/2}$

10

b) Here: $x_i \to y_i = x_i - (H^{-1})_{ij}$ and $y_i \to z_i = (U^{-1})_{ij} y_j$

According to (a), we choose $U^{-1} = U^{+}$ unitary.
Then

$$\left|\frac{\partial(x_1, x_1^*, \dots x_n, x_n^*)}{\partial(y_1, y_1^*, \dots y_n, y_n^*)}\right| = 1 \text{ and } \left|\frac{\partial(y_1, y_1^*, \dots)}{\partial(z_1, z_1^*, \dots)}\right| = \det U \cdot \det U^* = 1$$

Single-variable integration:

$$\int \frac{dz\,dz^*}{2\pi i} e^{-h|z|^2} = \int \frac{d(\mathrm{Re}\,z)\,d(\mathrm{Im}\,z)}{\pi} e^{-h(\mathrm{Re}\,z^2 + \mathrm{Im}\,z^2)} =$$

$$= \frac{1}{\sqrt{h}} \cdot \frac{1}{\sqrt{h}} = \frac{1}{h}$$

H hermitian $\Rightarrow H_{ij}^* = H_{ji}$ and $(H^{-1})_{ij}^* = (H^{-1})_{ji}$

Exponent:

$$-x_i^* H_{ij} x_j + j_i^* x_i + j_i x_i^* =$$

$$= -\left(y_i^* + (H^{-1})_{ik}^* j_k^*\right) H_{ij} \left(y_j + (H^{-1})_{jl} j_l\right) + j_i^* \left(y_i + (H^{-1})_{ij} j_j\right)$$

$$+ j_i \left(y_i^* + (H^{-1})_{ij}^* j_j^*\right) =$$

$$= -\Big[y_i^* H_{ij} y_j + y_i^* \cancel{\underbrace{H_{ij} (H^{-1})_{jl}}_{\delta_{il}}} j_l + j_k^* \cancel{\underbrace{(H^{-1})_{ki} H_{ij}}_{\delta_{kj}}} y_j$$

$$+ j_k^* (H^{-1})_{ki} H_{ij} (H^{-1})_{jl} j_l \Big] + \cancel{j_i^* y_i} + j_i^* (H^{-1})_{ij} j_j$$

$$+ \cancel{j_i y_i^*} + j_j^* (H^{-1})_{ji} j_i =$$

$$= -y_i^* H_{ij} y_j + j_i^* (H^{-1})_{ij} j_j$$

and

$$y_i^* H_{ij} y_j = U_{ik}^* z_k^* H_{ij} U_{jl} z_l =$$

$$= z_k^* (U^+)_{ki} H_{ij} U_{jl} z_l = z_k^* (U^+ H U)_{kl} z_l = \sum_i h_i |z_i|^2$$

choose U such that $U^+ H U = \tilde{H} = \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix}$

$$\Rightarrow \int \left(\prod_{i=1}^{n} \frac{dx_i^* dx_i}{2\pi i} \right) e^{-x_i^* H_{ij} x_j + J_i^* x_i + J_i x_i^*} =$$

$$= \left| \frac{\partial(x_1, x_1^*, \ldots)}{\partial(y_1, y_1^*, \ldots)} \right| \cdot \left| \frac{\partial(y_1, y_1^*, \ldots)}{\partial(z_1, z_1^*, \ldots)} \right| e^{J_i^* (H^{-1})_{ij} J_j} \prod_{i=1}^{n} \int \frac{dz_i^* dz_i}{2\pi i} e^{-h_i |z_i|^2}$$

$$= e^{J_i^* (H^{-1})_{ij} J_j} \prod_{i=1}^{n} \frac{1}{h_i} = \frac{e^{J_i^* (H^{-1}) J_j}}{\det H} \qquad \square$$

10

c) $$\int d\xi^* d\xi \, e^{-a \xi^* \xi} = \underbrace{\int d\xi^* d\xi}_{0} (1 - a \xi^* \xi) =$$

$$- a \int d\xi^* d\xi \, \xi^* \xi = a \int d\xi^* d\xi \, \xi \xi^* = a \quad ✓$$

Exponent:

$$-\eta_i^* H_{ij} \eta_j + \xi_i^* \eta_i + \eta_i^* \xi_i =$$

$$= -(\lambda_i^* + (H^{-1})_{ik}^* \xi_k^*) H_{ij} (\lambda_j + (H^{-1})_{jl} \xi_l) + \xi_i^* (\lambda_i + (H^{-1})_{im} \xi_m)$$

$$+ (\lambda_i^* + (H^{-1})_{iq}^* \xi_q^*) \xi_i =$$

$$= -\lambda_i^* H_{ij}\lambda_j - \xi_k^*(H^{-1})_{ki}\cancel{\overbrace{H_{ij}(H^{-1})_{jl}}^{\delta_{il}}}\xi_l - \lambda_i^*\cancel{\overbrace{H_{ij}(H^{-1})_{jl}}^{\delta_{il}}}\xi_l$$

$$- \xi_k^*\cancel{\underbrace{(H^{-1})_{ki}H_{ij}}_{\delta_{kj}}}\lambda_j + \cancel{\xi_i^*\lambda_i} + \xi_i^*(H^{-1})_{im}\xi_m$$

$$+ \cancel{\lambda_i^*\xi_i} + \cancel{\xi_q^*(H^{-1})_{qi}}\xi_i$$

$$= -\lambda_i^* H_{ij}\lambda_j + \xi_i^*(H^{-1})_{ij}\xi_j$$

Here also $\left|\dfrac{\partial(\lambda_1,\lambda_1^*,\ldots)}{\partial(\eta_1,\eta_1^*\ldots)}\right| = \left|\dfrac{\partial(\mu_1,\mu_1^*,\ldots)}{\partial(\lambda_1,\lambda_1^*\ldots)}\right| = 1$

$$\Rightarrow \int\left(\prod_{i=1}^n d\eta_i^* d\eta_i\right) e^{-\eta_i^* H_{ij}\eta_j + \xi_i^*\eta_i + \eta_i^*\xi_i} =$$

$$= \int\left(\prod_{i=1}^n d\lambda_i^* d\lambda_i\right) e^{-\lambda_i^* H_{ij}\lambda_j} \cdot e^{\xi_i^*(H^{-1})_{ij}\xi_j} =$$

$$= e^{\xi_i^*(H^{-1})_{ij}\xi_j}\underbrace{\prod_{i=1}^n \underbrace{\int d\mu_i^* d\mu_i\, e^{-h_i\mu_i^*\mu_i}}_{h_i}}_{\det H}$$

$$= \boxed{\det H\, e^{\xi_i^*(H^{-1})_{ij}\xi_j}} \qquad \square$$

20

3.) The easiest way is to use a^+, a:

$$H = \hbar\omega(a^+a + \tfrac{1}{2}) = \epsilon(a^+a + \tfrac{1}{2})$$

$$Z(t) = \int_{\alpha_f^*=\alpha_i} \frac{d\alpha_f^* d\alpha_f}{2\pi i} \int \prod_{i=1}^{M-1} \frac{d\alpha_i^* d\alpha_i}{2\pi i} \, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[i\hbar \alpha^*(t)\dot{\alpha}(t) - H[\alpha^*(t),\alpha(t)]\right]}$$

$$\underset{=}{\text{discretized}} \int \prod_{k=1}^{M} \frac{d\alpha_k^* d\alpha_k}{2\pi i} \, e^{\frac{i}{\hbar}\delta t \sum_{k=1}^{M}\left[i\hbar \alpha_k^* \frac{\alpha_k - \alpha_{k-1}}{\delta t} - \epsilon(\alpha_k^* \alpha_{k-1} + \frac{1}{2})\right]}$$

Exponent for $\delta t \to \delta\tau = i\delta t = \frac{\hbar\beta}{M} \Rightarrow \boxed{\delta t = -i\hbar\frac{\beta}{M}}$

$$\to \frac{\beta}{M}\sum_{k=1}^{M}\left[-\frac{M}{\beta}\alpha_k^*(\alpha_k - \alpha_{k-1}) - \epsilon(\alpha_k^*\alpha_{k-1} + \tfrac{1}{2})\right]$$

$$= -\sum_{k=1}^{M}\left[\alpha_k^*\alpha_k - \alpha_k^*\alpha_{k-1} + \frac{\beta}{M}\epsilon(\alpha_k^*\alpha_{k-1} + \tfrac{1}{2})\right]$$

$$\overset{!}{=} -\alpha_k^* \sigma_{k\ell}\alpha_\ell - \underbrace{\sum_{k=1}^{M}\frac{\beta}{M}\frac{\epsilon}{2}}_{\frac{\beta\epsilon}{2}}$$

$$\Rightarrow \sigma_{kk} = 1, \quad \sigma_{k,k-1} = -\underbrace{\left(1 - \frac{\beta\epsilon}{M}\right)}_{a} = \sigma_{1,M}, \quad \text{all others} = 0$$

$$\det\sigma = \det\begin{pmatrix} 1 & & & -a \\ -a & \ddots & & \\ & \ddots & \ddots & \\ & & -a & 1 \end{pmatrix} = 1\cdot 1 + (-1)^{M-1}(-a)(-a)^{M-1}$$

$$\Rightarrow \det\sigma = \lim_{M\to\infty} 1 - (1 - \tfrac{\beta}{M}\epsilon)^M = 1 - e^{-\beta\epsilon}$$

$$\Rightarrow Z(\beta) = e^{-\frac{\beta\epsilon}{2}}\int \prod_{k=1}^{M}\frac{d\alpha_k^* d\alpha_k}{2\pi i}\, e^{-\alpha_i^*\sigma_{ij}\alpha_j} = \frac{e^{-\frac{\beta\epsilon}{2}}}{1 - e^{-\beta\epsilon}} =$$

$$= \frac{1}{e^{\frac{\beta\epsilon}{2}} - e^{-\frac{\beta\epsilon}{2}}} = \frac{1}{2\sinh\frac{\beta\epsilon}{2}}$$

$$\Rightarrow Z(t) = \frac{1}{2\sinh\frac{\hbar\omega}{2}\cdot\frac{i}{\hbar}t} = \frac{1}{2i\sin\frac{\omega t}{2}} \qquad \square$$

A solution using a Fourier transform directly of the x-p version of the Hamiltonian can be found in many books, e.g., "QM and Path Integrals" by Feynman, Hibbs (McGraw-Hill)