RADIX-2 FAST FOURIER TRANSFORM

http://www.phys.uconn.edu/~rozman/Courses/m3511_18s/

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The discrete Fourier transform (DFT) of the input $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1})$ is defined as following:

$$X_k = \sum_{n=0}^{N-1} x_n \, e^{-\frac{2\pi i}{N} n k},$$

where $k = 0, 1, \ldots, N-1$.

Radix-2 algorithm first computes the DFTs of the even-indexed inputs $(x_0, x_2, \ldots, x_{N-2})$ and of the odd-indexed inputs $(x_1, x_3, \ldots, x_{N-1})$, and then combines those two results to produce the DFT of the whole sequence. This idea can then be performed recursively to reduce the overall runtime from $O(N^2)$ to $O(N \log N)$. This simplified form assumes that N is a power of two; since the number of sample points N can usually be chosen freely by the application, this is often not an important restriction.

Lets rearrange the DFT of $\mathbf{x}$ into two parts: a sum over the even-numbered indices and a sum over the odd-numbered indices:

$$X_k = \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N}(2m)k} + \sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N}(2m+1)k}.$$

One can factor a common multiplier $e^{-\frac{2\pi i}{N} k}$ out of the second sum. It is then clear that the two sums are the DFT of the even-indexed part and the DFT of odd-indexed part of x_n. Denote the DFT of the even-indexed inputs by E_k and the DFT of the odd-indexed inputs by O_k and we obtain:

$$X_k = \underbrace{\sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N/2}mk}}_{\text{DFT of even-indexed part}} + e^{-\frac{2\pi i}{N}k} \underbrace{\sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N/2}mk}}_{\text{DFT of odd-indexed part}} = E_k + e^{-\frac{2\pi i}{N}k} O_k.$$

E_k and O_k are periodic with the period $N/2$,

$$E_{k+\frac{N}{2}} = E_k$$

and

$$O_{k+\frac{N}{2}} = O_k.$$

Therefore, we can rewrite the above equation as

$$X_k = \begin{cases} E_k + e^{-\frac{2\pi i}{N}k} O_k & \text{for } 0 \le k < N/2, \\ E_{k-N/2} + e^{-\frac{2\pi i}{N}k} O_{k-N/2} & \text{for } N/2 \le k < N, \end{cases}$$

where we used the periodicity of O_k and E_k to translate the index k.

Now using the following relation of the twiddle factor $e^{-2\pi ik/N}$,

$$e^{\frac{-2\pi i}{N}(k+N/2)} = e^{\frac{-2\pi ik}{N} - \pi i} = e^{-\pi i} e^{\frac{-2\pi ik}{N}} = -e^{\frac{-2\pi ik}{N}}$$

we can rewrite X_k as:

$$\begin{aligned} X_k &= E_k + e^{-\frac{2\pi i}{N}k} O_k, \\ X_{k+\frac{N}{2}} &= E_k - e^{-\frac{2\pi i}{N}k} O_k. \end{aligned}$$

This result, expressing the DFT of length N recursively in terms of two DFTs of size $N/2$, is the core of the radix-2 fast Fourier transform.