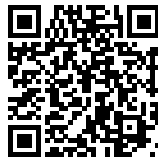


## RADIX-2 FAST FOURIER TRANSFORM

[http://www.phys.uconn.edu/~rozman/Courses/m3511\\_18s/](http://www.phys.uconn.edu/~rozman/Courses/m3511_18s/)



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The discrete Fourier transform (DFT) of the input  $\mathbf{x} = (x_0, x_1, \dots, x_{N-1})$  is defined as following:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N}nk},$$

where  $k = 0, 1, \dots, N-1$ .

Radix-2 algorithm first computes the DFTs of the even-indexed inputs  $(x_0, x_2, \dots, x_{N-2})$  and of the odd-indexed inputs  $(x_1, x_3, \dots, x_{N-1})$ , and then combines those two results to produce the DFT of the whole sequence. This idea can then be performed recursively to reduce the overall runtime from  $O(N^2)$  to  $O(N \log N)$ . This simplified form assumes that  $N$  is a power of two; since the number of sample points  $N$  can usually be chosen freely by the application, this is often not an important restriction.

Lets rearrange the DFT of  $\mathbf{x}$  into two parts: a sum over the even-numbered indices and a sum over the odd-numbered indices:

$$X_k = \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N}(2m)k} + \sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N}(2m+1)k}.$$

One can factor a common multiplier  $e^{-\frac{2\pi i}{N}k}$  out of the second sum. It is then clear that the two sums are the DFT of the even-indexed part and the DFT of odd-indexed part of  $x_n$ . Denote the DFT of the even-indexed inputs by  $E_k$  and the DFT of the odd-indexed inputs

by  $O_k$  and we obtain:

$$X_k = \underbrace{\sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N/2} mk}}_{\text{DFT of even-indexed part}} + e^{-\frac{2\pi i}{N} k} \underbrace{\sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N/2} mk}}_{\text{DFT of odd-indexed part}} = E_k + e^{-\frac{2\pi i}{N} k} O_k.$$

$E_k$  and  $O_k$  are periodic with the period  $N/2$ ,

$$E_{k+\frac{N}{2}} = E_k$$

and

$$O_{k+\frac{N}{2}} = O_k.$$

Therefore, we can rewrite the above equation as

$$X_k = \begin{cases} E_k + e^{-\frac{2\pi i}{N} k} O_k & \text{for } 0 \leq k < N/2, \\ E_{k-N/2} + e^{-\frac{2\pi i}{N} k} O_{k-N/2} & \text{for } N/2 \leq k < N, \end{cases}$$

where we used the periodicity of  $O_k$  and  $E_k$  to translate the index  $k$ .

Now using the following relation of the twiddle factor  $e^{-2\pi i k/N}$ ,

$$e^{-\frac{2\pi i}{N}(k+N/2)} = e^{-\frac{2\pi i k}{N} - \pi i} = e^{-\pi i} e^{-\frac{2\pi i k}{N}} = -e^{-\frac{2\pi i k}{N}}$$

we can rewrite  $X_k$  as:

$$\begin{aligned} X_k &= E_k + e^{-\frac{2\pi i}{N} k} O_k, \\ X_{k+\frac{N}{2}} &= E_k - e^{-\frac{2\pi i}{N} k} O_k. \end{aligned}$$

This result, expressing the DFT of length  $N$  recursively in terms of two DFTs of size  $N/2$ , is the core of the radix-2 fast Fourier transform.