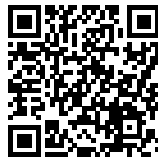


ORTHOGONAL TRAJECTORIES

http://www.phys.uconn.edu/~rozman/Courses/m3410_18s/



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1 Families of curves

We have seen that the general solution of a first order differential equation normally contains one arbitrary constant, called a parameter. When this parameter is assigned various values, we obtain a one-parameter family of curves. Each of these curves is a particular solution, or integral curve, of the given differential equation, and all of them together constitute its general solution.

Conversely, as we might expect, the curves of any one-parameter family are integral curves of some first order differential equation. If the family is

$$f(x, y, c) = 0, \tag{1}$$

then its differential equation can be found by the following steps. First, differentiate Eq. (1) implicitly with respect to x to get a relation of the form

$$g\left(x, y, \frac{dy}{dx}, c\right) = 0. \tag{2}$$

Next, eliminate the parameter c from Eq. (1) and Eq. (2) to obtain

$$F\left(x, y, \frac{dy}{dx}\right) = 0 \tag{3}$$

as the desired differential equation.

For example,

$$x^2 + y^2 = c^2 \quad (4)$$

is the equation of the family of circles of radius c with centers at the origin. On differentiation with respect to x this becomes

$$2x + 2y \frac{dy}{dx} = 0.$$

Since c is already absent, there is no need to eliminate it and

$$y \frac{dy}{dx} + x = 0 \quad (5)$$

is the differential equation of the given family of circles.

Similarly,

$$x^2 + y^2 = 2cx \quad (6)$$

is the equation of the family of all circles tangent to the y -axis at the origin. Indeed, we can rewrite Eq. (6) as following

$$(x^2 - 2cx + c^2) + y^2 = c^2,$$

or

$$(x - c)^2 + y^2 = c^2,$$

which is the equation of the family of circles of radius c with centers on the x axis at $x = c$.

When we differentiate Eq. (6) with respect to x , we obtain

$$2x + 2y \frac{dy}{dx} = 2c,$$

or

$$x + y \frac{dy}{dx} = c. \quad (7)$$

The parameter c is still present, so it is necessary to eliminate it by combining Eq. (6) and Eq. (7). This yields

$$x + y \frac{dy}{dx} = \frac{x^2 + y^2}{2x},$$

or

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}. \quad (8)$$

as the differential equation of the family Eq. (6).

2 Orthogonal trajectories

As an interesting application of these procedures, we consider the problem of finding orthogonal trajectories. To explain what this problem is, we observe that the family of circles represented by Eq. (4) and the family $y = mx$ of straight lines through the origin have the following property: each curve in either family is orthogonal (i.e., perpendicular) to every curve in the other family. Whenever two families of curves are related in this way, each is said to be a family of orthogonal trajectories of the other. Orthogonal trajectories are of interest in the geometry of plane curves, and also in certain parts of applied mathematics. For instance, if an electric current is flowing in a plane sheet of conducting material, then the lines of equal potential are the orthogonal trajectories of the lines of current flow.

In the example of the circles centered on the origin, it is geometrically obvious that the orthogonal trajectories are the straight lines through the origin, and conversely. In order to cope with more complicated situations, however, we need an analytic method for finding orthogonal trajectories.

Suppose that

$$\frac{dy}{dx} = f(x, y) \quad (9)$$

is the differential equation of the family of curves. These curves are characterized by the fact that at any point (x, y) on any one of them the slope is given by $f(x, y)$. The orthogonal trajectory through the same point, has as its slope the negative reciprocal of the first slope. Thus, along any orthogonal trajectory, we have

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}$$

or

$$-\frac{dx}{dy} = f(x, y). \quad (10)$$

Our method of finding the orthogonal trajectories of a given family of curves is therefore as follows: first, find the differential equation of the family; next, replace dy/dx by dx/dy to obtain the differential equation of the orthogonal trajectories; and finally, solve this new differential equation.

If we apply this method to the family of circles Eq. (4) with differential equation (5), we get

$$-y \frac{dx}{dy} + x = 0,$$

or

$$\frac{dy}{dx} = \frac{y}{x} \tag{11}$$

as the differential equation of the orthogonal trajectories. We can now separate the variables in Eq. (11) to obtain

$$\frac{dy}{y} = \frac{dx}{x},$$

which on direct integration yields

$$\log y = \log x + \log c$$

or

$$y = cx$$

as the equation of the orthogonal trajectories.

References

- [1] George F. Simmons, *Differential Equations with Applications and Historical Notes*, 3rd ed., Chapman and Hall/CRC, 2017