MATH 3410  
HW 3  
Due: Wed Feb 7, 2018

Name: ____________________________

Date: ____________________________

Collaborators: ____________________________

(Collaborators submit their individually written assignments together)

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Instructor/grader comments:
Modeling with first order differential equations

1. The energy of the harmonic oscillator (e.g. a mass attached to a linear spring):

\[ E = \frac{m}{2} \left( \frac{dx}{dt} \right)^2 + \frac{k}{2} x^2, \]

where \( m \) is the mass, \( m > 0 \), \( k \) is the spring elastic constant, \( k > 0 \), \( x \) it the coordinate of the mass measured from its equilibrium position, \( \frac{dx}{dt} \) is the velocity.

Find the period of the oscillations vs oscillator energy.

(a) (10 points) Use the expression for the energy above to find the first order differential equation for \( x(t) \).

(b) (5 points) The oscillator moves between two turning points, \( x_\pm \) (\( x_- < 0 \), \( x_+ > 0 \)) such that \( v(x_\pm) = 0 \).

Use the expression for the oscillator’s energy to find the coordinates of the turning points.

(c) (5 points) Introduce the new dimensionless dependent variable, \( u \sim x \), such that \(-1 \leq u \leq 1\). Write the differential equation for \( u(t) \).

(d) (10 points) Find the general solution of the equation in the form \( t = f(u) \)

Hint:

\[ \int \frac{du}{\sqrt{1-u^2}} = \arcsin u \]

(e) (10 points) Use the general solution in part (d) to find the period of the oscillations.

Hint: for simplicity find the time to move from \( u = -1 \) to \( u = 1 \). The period of the oscillator is double this time.

Orthogonal trajectories

2. Find the orthogonal trajectories of the family of ellipses

\[ x^2 + 2y^2 = c^2 \]

(a) (10 points) Find the differential equation for the family of ellipses above

(b) (10 points) Find the differential equation for the ‘orthogonal’ curves
(c) (15 points) Find the general solution of the equation you derived in part (b). On the same graph plot the family of the ellipses and the family of their orthogonal trajectories. Attach the printout of your graph and the code you used to plot.

**Autonomous equations**

3. A chemical reaction involves the interaction of one molecule of a substance $P$ with one molecule of a substance $Q$ to produce one molecule of a new substance $X$. Suppose that $p$ and $q$ are are the initial concentrations of $P$ and $Q$ respectively. Let $x(t)$ be the concentration of $X$ at time $t$, $x(0) = 0$. Then $p - x(t)$ and $q - x(t)$ are the concentrations of $P$ and $Q$ at time $t$. The rate at which the reaction occurs is given by the equation

$$\frac{dx}{dt} = \alpha(p - x)(q - x),$$

where $\alpha$ is a positive constant.

(a) (10 points) Assuming that $p \neq q$ determine the limiting value of $x(t)$ as $t \to \infty$ without solving the differential equation.

(b) (15 points) If the substances $P$ and $Q$ are the same, then $p = q$ and the equation for $x(t)$ is replaced by

$$\frac{dx}{dt} = \alpha(p - x)^2.$$

Determine the limiting value of $x$ without solving the equation. Then solve the initial value problem and determine $x(t)$ for any $t$. 

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