BESSEL FUNCTIONS

http://www.phys.uconn.edu/~rozman/Courses/m3410_18s/

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Bessel functions are the canonical solutions of Bessel's differential equation:

$$x^2 y'' + x y' + \left(x^2 - \nu^2\right) y = 0 \tag{1}$$

for an arbitrary complex number ν. The parameter ν is called the order of the Bessel function. Bessel's equation arises when finding solutions to Laplace's equation and the Helmholtz equation in cylindrical or spherical coordinates.

1 Frobenius series solution

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}. \tag{2}$$

Derivatives and products, for the reference:

$$x y(x) = \sum_{n=0}^{\infty} a_n x^{r+n+1} = \sum_{m=1}^{\infty} a_{m-1} x^{r+m} = \sum_{n=1}^{\infty} a_{n-1} x^{r+n} \tag{3}$$

$$x^2 y(x) = \sum_{n=0}^{\infty} a_n x^{r+n+2} = \sum_{m=2}^{\infty} a_{m-2} x^{r+m} = \sum_{n=2}^{\infty} a_{n-2} x^{r+n}. \tag{4}$$

$$
\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty}(r+n)\,a_n x^{r+n-1} = \sum_{m=-1}^{\infty}(r+m+1)\,a_{m+1}x^{r+m} \\
&= \sum_{n=-1}^{\infty}(r+n+1)\,a_{n+1}x^{r+n} && (5) \\
x\,y'(x) &= \sum_{n=0}^{\infty}(r+n)\,a_n x^{r+n}, && (6) \\
x^2\,y'(x) &= \sum_{n=0}^{\infty}(r+n)\,a_n x^{r+n+1} = \sum_{m=1}^{\infty}(r+m-1)\,a_{m-1}x^{r+m} \\
&= \sum_{n=1}^{\infty}(r+n-1)\,a_{n-1}x^{r+n}. && (7)
\end{aligned}
$$

$$
\begin{aligned}
y''(x) &= \sum_{n=0}^{\infty}(r+n)(r+n-1)\,a_n x^{r+n-2} \\
&= \sum_{m=-2}^{\infty}(r+m+2)(r+m+1)\,a_{m+2}x^{r+m} \\
&= \sum_{n=-2}^{\infty}(r+n+2)(r+n+1)\,a_{n+2}x^{r+n}, && (8) \\
x\,y''(x) &= \sum_{n=0}^{\infty}(r+n)(r+n-1)\,a_n x^{r+n-1} \\
&= \sum_{m=-1}^{\infty}(r+m+1)(r+m)\,a_{m+1}x^{r+m} \\
&= \sum_{n=-1}^{\infty}(r+n+1)(r+n)\,a_{n+1}x^{r+n}, && (9) \\
x^2\,y''(x) &= \sum_{n=0}^{\infty}(r+n)(r+n-1)\,a_n x^{r+n}. && (10)
\end{aligned}
$$

2 Bessel equation

$$x^2 y'' + x y' + \left(x^2 - \nu^2\right) y = 0. \tag{11}$$

$$x^2 y'' + x y' - \nu^2 y + x^2 y = 0. \tag{12}$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)\, a_n x^{r+n} + \sum_{n=0}^{\infty}(r+n)\, a_n x^{r+n} - \nu^2 \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} = 0 \tag{13}$$

$$\begin{aligned}
& r(r-1)\, a_0 x^r + r(r+1)\, a_1 x^{r+1} + \sum_{n=2}^{\infty}(r+n)(r+n-1)\, a_n x^{r+n} \\
& + r\, a_0 x^r + (r+1)\, a_1 x^{r+1} + \sum_{n=2}^{\infty}(r+n)\, a_n x^{r+n} \\
& -\nu^2 a_0 x^r - \nu^2 a_1 x^{r+1} - \nu^2 \sum_{n=2}^{\infty} a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} = 0
\end{aligned} \tag{14}$$

$$\begin{aligned}
& a_0 x^r \left(r(r-1) + r - \nu^2\right) + a_1 x^{r+1}\left(r(r+1) + (r+1) - \nu^2\right) \\
& + \sum_{n=2}^{\infty} x^{r+n} \left\{ \left[(r+n)(r+n-1) + (r+n) - \nu^2\right] a_n + a_{n-2} \right\} \quad = \quad 0
\end{aligned} \tag{15}$$

$$\begin{aligned}
& a_0 x^r \left(r^2 - \nu^2\right) + a_1 x^{r+1}\left((r+1)^2 - \nu^2\right) \\
& + \sum_{n=2}^{\infty} x^{r+n} \left\{ \left[(r+n)^2 - \nu^2\right] a_n + a_{n-2} \right\} \quad = \quad 0
\end{aligned} \tag{16}$$

$$a_0\left(r^2 - \nu^2\right) = 0 \quad \longrightarrow \quad a_0 \neq 0, \quad r_{1,2} = \pm\nu, \tag{17}$$

$$a_1\left((r+1)^2 - \nu^2\right) = 0 \quad \longrightarrow \quad a_1 = 0. \tag{18}$$

$$a_n = -\frac{a_{n-2}}{(r+n+\nu)(r+n-\nu)}. \tag{19}$$

Hence,

$$a_3 = a_5 = \ldots = a_{2n+1} \ldots = 0. \tag{20}$$

$$a_{2n} = -\frac{a_{2n-2}}{2^2 n(n+\nu)}. \tag{21}$$

The choice of a_0 is arbitrary. By convention, for integer $\nu = m$,

$$a_0 = \frac{1}{2^m m!} \tag{22}$$

2.1 Bessel equation of order one-half

For $\nu = \frac{1}{2}$,

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0. \tag{23}$$

The largest solution of the indicial equation Eq. (17) is $r = \frac{1}{2}$, and Eq. (21) gives:

$$a_{2n} = -\frac{a_{2n-2}}{2n(2n+1)}. \tag{24}$$

$$a_2 = -\frac{a_0}{2\cdot 3} = -\frac{a_0}{3!}, \quad a_4 = -\frac{a_2}{4\cdot 5} = \frac{a_0}{2\cdot 3\cdot 4\cdot 5}, \quad a_6 = -\frac{a_4}{6\cdot 7} = -\frac{a_0}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}, \tag{25}$$

The general term,

$$a_{2n} = \frac{(-1)^n a_0}{(2n+1)!}, \quad n = 0, 1, 2, \ldots \tag{26}$$

Taking $a_0 = 1$,

$$y_1(x) = x^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = x^{-\frac{1}{2}} \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{\sin x} = \frac{1}{\sqrt{x}} \sin x. \tag{27}$$

The canonical solution,

$$J_{\frac{1}{2}}(x) \equiv \sqrt{\frac{2}{\pi}}\, y_1(x) = \sqrt{\frac{2}{\pi x}} \sin x. \tag{28}$$

Wronskian of Eq. (23)

$$W(x) = \exp\left(-\int \frac{\mathrm{d}x}{x}\right) = \frac{1}{x}. \tag{29}$$

The second solution,

$$y_2(x) = y_1(x) \int \frac{W(x)}{y_1^2(x)}\,\mathrm{d}x = \frac{\sin x}{\sqrt{x}} \int \frac{\mathrm{d}x}{\sin^2 x} = \frac{\sin x}{\sqrt{x}}\,\frac{\cos x}{\sin x} = \frac{1}{\sqrt{x}}\cos x. \tag{30}$$

The canonical solution,

$$J_{-\frac{1}{2}}(x) \equiv \sqrt{\frac{2}{\pi}}\,y_2(x) = \sqrt{\frac{2}{\pi x}}\cos x. \tag{31}$$

2.2 Bessel equation of order zero

$$x^2 y'' + x y' + x^2 y = 0. \tag{32}$$

$$\nu = 0 \quad \longrightarrow \quad r = 0. \tag{33}$$

$$a_{2n} = -\frac{a_{2n-2}}{2^2 n^2}, \quad n = 1, 2, \ldots \tag{34}$$

$$a_2 = -\frac{a_0}{2^2}, \quad a_4 = -\frac{a_2}{2^2\,2^2} = \frac{a_0}{2^4\,2^2}, \quad a_6 = -\frac{a_4}{2^2\,3^2} = -\frac{a_0}{2^6\,(2\cdot 3)^2}, \tag{35}$$

$$a_8 = -\frac{a_6}{2^2\,4^2} = \frac{a_0}{2^8\,(2\cdot 3\cdot 4)^2}, \quad a_{10} = -\frac{a_8}{2^2\,5^2} = -\frac{a_0}{2^{10}\,(2\cdot 3\cdot 4\cdot 5)^2}, \quad \ldots \tag{36}$$

The general term

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n}(n!)^2}. \tag{37}$$

Chosing per Eq. (22) $a_0 = 0$,

$$y_1(x) = J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}. \tag{38}$$

We search for the second solution, $y_2(x)$, in the following form:

$$y_2(x) = J_0(x)\ln(x) + \sum_n^{\infty} b_n x^n \tag{39}$$

$$\Big(J_0(x)\ln(x)\Big)' = J_0'(x)\ln(x) + \frac{J_0(x)}{x} \tag{40}$$

$$\begin{aligned}\Big(J_0(x)\ln(x)\Big)'' &= J_0''(x)\ln(x) + \frac{J_0'(x)}{x} + \frac{J_0'(x)}{x} - \frac{J_0(x)}{x^2} \quad (41)\\ &= J_0''(x)\ln(x) + 2\,\frac{J_0'(x)}{x} - \frac{J_0(x)}{x^2} \quad (42)\end{aligned}$$

$$\begin{aligned}x^2\Big(J_0(x)\ln(x)\Big)'' + x\Big(J_0(x)\ln(x)\Big)' + x^2 J_0(x)\ln(x) &= \quad (43)\\ \underbrace{\Big(x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x)\Big)}_{0}\ln(x) + 2\,x J_0'(x) - J_0(x) + J_0(x) &= 2\,x J_0'(x) \quad (44)\end{aligned}$$

$$2\,x J_0'(x) = 2\,x\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right)' = 2\,x\sum_{n=1}^{\infty}\frac{(-1)^n 2n\,x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2(n-1)}n!(n-1)!}. \tag{45}$$

The summation in Eq. (45) is over even powers of x, starting from x^2. Therefore the summation in the "counterterm" $\sum b_n x^n$ must be over even n ($b_1 = b_3 = \ldots = 0$) and $b_0 = 0$. Using Eq. (45) and Eq. (16),

$$\sum_{n=1}^{\infty} x^{2n}\Big\{(2n)^2 b_{2n} + b_{2(n-1)}\Big\} = -\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2(n-1)}n!(n-1)!}, \tag{46}$$

$$(2n)^2 b_{2n} + b_{2(n-1)} = -\frac{(-1)^n}{2^{2(n-1)}n!(n-1)!}, \tag{47}$$

$$b_2 = \frac{1}{2^2}, \quad b_4 = -\frac{1}{2^2\,4^2}\left(1+\frac{1}{2}\right), \quad b_6 = -\frac{1}{2^2\,4^2\,6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right). \tag{48}$$

The general term,

$$b_{2m} = \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2}, \tag{49}$$

where

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}. \tag{50}$$

$$y_2(x) = J_0(x) \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n} H_n}{2^{2n}(n!)^2}. \tag{51}$$

Bessel function of the second kind of order zero,

$$Y_0(x) = \frac{2}{\pi} \left[y_2(x) + (\gamma - \ln 2) J_0(x) \right], \tag{52}$$

where $\gamma \approx 0.577$ is known as Euler constant.

3 Three term recurrence relation for Bessel functions

$$x^2 J''_{n+1} + x J'_{n+1} + \left(x^2 - (n+1)^2\right) J_{n+1} = 0. \tag{53}$$

$$x^2 J''_{n-1} + x J'_{n-1} + \left(x^2 - (n-1)^2\right) J_{n-1} = 0. \tag{54}$$

$$y_n(x) \equiv J_{n-1}(x) + J_{n+1}(x) \tag{55}$$

$$x^2 y''_n + x y'_n + \left(x^2 - n^2 - 1\right) y_n = 0. \tag{56}$$

$$y_n(x) \sim \frac{J_n(x)}{x}. \tag{57}$$

$$\left(\frac{J_n}{x}\right)' = \frac{J'_n}{x} - \frac{J_n}{x^2} \tag{58}$$

$$\left(\frac{J_n}{x}\right)'' = \frac{J''_n}{x} - \frac{2J'_n}{x^2} + \frac{2J_n}{x^3} \tag{59}$$

$$
\begin{aligned}
x^2\left(\frac{J_n}{x}\right)'' + x\left(\frac{J_n}{x}\right)' + (x^2-n^2-1)\frac{J_n}{x} &= xJ_n'' - 2J_n' + \frac{2J_n}{x} + J_n' - \frac{J_n}{x} + (x^2-n^2)\frac{J_n}{x} - \frac{J_n}{x} \\
&= \frac{1}{x}\underbrace{\left(x^2J_n'' - xJ_n' + (x^2-n^2)J_n\right)}_{0} = 0 \qquad (60)
\end{aligned}
$$

$$
C(n)\,\frac{J_n(x)}{x} = J_{n-1}(x) + J_{n+1}(x) \qquad (61)
$$

For $x \ll 1$,

$$
J_n(x) \approx a_0 x^r = \frac{x^n}{2^n n!} \qquad (62)
$$

$$
C(n) = \lim_{x\to 0} \frac{x(J_{n-1}(x) + J_{n+1}(x))}{J_n(x)} = \frac{\dfrac{x^n}{2^{n-1}(n-1)!}}{\dfrac{x^n}{2^n n!}} = 2n \qquad (63)
$$

$$
\frac{2n\,J_n(x)}{x} = J_{n-1}(x) + J_{n+1}(x) \qquad (64)
$$

The result is valid also for non-integer order ν.

$$
\frac{2\nu\,J_\nu(x)}{x} = J_{\nu-1}(x) + J_{\nu+1}(x) \qquad (65)
$$