THINGS TO KNOW FOR MIDTERM I

SPRING SEMESTER 2017

http://www.phys.uconn.edu/~rozman/Courses/P2400_17S/

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Gamma function:

\[ \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt \]
\[ \Gamma(x + 1) = x \Gamma(x) \]
\[ \Gamma(n) = (n - 1)! \quad n = 1, 2, 3, \ldots \]
\[ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(1) = 1 \]

Beta function:

\[ B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} \, dt \]
\[ B(x, y) = B(y, x) \]
\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \]

Page 1 of 3
Leibniz's formula:

\[
\frac{d}{dx} \left\{ \int_{a(x)}^{b(x)} f(t, x) \, dt \right\} = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(t, x) \, dt + f(b(x), x) \frac{db}{dx} - f(a(x), x) \frac{da}{dx}
\]

Euler's formula:

\[
e^{ix} = \cos(x) + i \sin(x), \quad \text{where} \quad i \equiv \sqrt{-1}
\]

\[
\cos x = \frac{1}{2} \left( e^{ix} + e^{-ix} \right)
\]

\[
\sin x = \frac{1}{2i} \left( e^{ix} - e^{-ix} \right)
\]

\[
e^{\pm i\pi} = -1, \quad e^{\pm i\frac{\pi}{2}} = \pm i, \quad e^{2\pi in} = 1, \quad n \in \mathbb{Z}
\]

Complex numbers – coordinate and polar form:

\[
z = x + iy = r e^{i\varphi}, \quad \text{where} \quad i \equiv \sqrt{-1} = e^{i\frac{\pi}{2}}
\]

\[
r \equiv |z| = \sqrt{x^2 + y^2}, \quad \tan \varphi = \frac{y}{x}, \quad \tan \frac{\varphi}{2} = \frac{y}{x + r} \quad \iff \quad x = r \cos \varphi, \quad y = r \sin \varphi
\]

Cauchy-Riemann equations:

\[
f(z) = u(x, y) + iv(x, y)
\]

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0
\]

Cauchy's integral theorem:

\[
\oint_C f(z) \, dz = 0,
\]

if \( f(z) \) is analytic inside a closed contour \( C \). Alternatively,

\[
\int_{L_1} f(z) \, dz = \int_{L_2} f(z) \, dz,
\]

where \( L_1 \) and \( L_2 \) are two different complex paths that share the start and the end points.
Laurent series:
\[ f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \ldots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \ldots, \]
\[ f(z) = \sum_{n=-m}^{\infty} a_n(z-z_0)^n. \]

\( f(z) \) has an isolated singularity at \( z_0 \); \( m \) - the order of the pole.
\[ a_n = \frac{1}{(m+n)!} \left. \frac{d^{m+n}}{dz^{m+n}} [(z-z_0)^m f(z)] \right|_{z=z_0}. \]

Residues:
\[ \oint_C f(z) \, dz = 2\pi i a_{-1}, \]
where \( C \) is a closed contour that includes \( z_0 \), the single singularity of \( f(z) \).

For a simple pole,
\[ \text{Res}(f(z), z = z_0) \equiv a_{-1}. \]

If \( f(z) = \frac{p(z)}{q(z)} \) then \( q(z_0) = 0 \) and
\[ \text{Res}(f(z), z = z_0) = \frac{p(z_0)}{q'(z_0)}, \]
where \( q' = \frac{dq}{dz}. \)

For a pole of order \( m \),
\[ \text{Res}(f(z), z = z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)]. \]

**Jordan's lemma:** yields a simple way to calculate the integral along the real axis of functions \( f(z) = e^{iad} g(z). \)