LAPLACE’S METHOD FOR INTEGRALS

LECTURE NOTES, SPRING SEMESTER 2017

http://www.phys.uconn.edu/~rozman/Courses/P2400_17S/

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Laplace’s method is a general technique for obtaining the asymptotic behavior of integrals in which the large parameter $\lambda$, $\lambda \to \infty$, appears in the exponent:

$$ I(\lambda) = \int_a^b f(t) e^{\lambda \phi(t)} \, dt = \int_a^b f(t) \left( e^{\phi(t)} \right)^\lambda \, dt. $$ (1)

Here $f(t)$ and $\phi(t)$ are real continuous functions, independent of $\lambda$. Integrals of this form are called Laplace integrals. Laplace’s method relies on the following observation: if the real continuous function $\phi(t)$ has its maximum on the interval $a \leq t \leq b$ at $t = t_0$ and if $f(t_0) \neq 0$, then it is only the immediate neighborhood of $t = t_0$ that contributes to the asymptotic expansion of $I(\lambda)$ for large $\lambda$.

Indeed, we can always write $e^{\phi(t)}$ as $e^{\phi(t_0)} e^{\psi(t)}$, where $e^{\phi(t_0)}$ is just a constant multiplication factor that can be factored out of the integral. Here we defined $\psi(t) \equiv \phi(t) - \phi(t_0)$. The maximal value of $\psi(t)$ is zero, thus the maximal value of $e^{\psi(t)}$ is one. A typical behavior of $e^{\psi(t)}$ is sketched in Fig. 1 in solid line. As we rise $e^{\psi(t)}$ into power $\lambda$, its maximum stays “fixed” at $(x_0, 1)$ but its wings are “moving down” toward the $x$ axis, thus making the graph narrower (see Fig. 1). Therefore we can replace $f(t)$ and $\phi(t)$ with their approximations that need to be good ones only in the vicinity of $t_0$.

The logic of the Laplace method works without changes for a more general form of the
Figure 1: Changes of the integrand in Laplace integral as the parameter $\lambda$ is increasing.

The integrand:

$$\int_a^b f(t) \left( \kappa(t) \right)^\lambda \, dt,$$

where $\lambda$ is a large parameter as before, $\lambda \to \infty$, $\kappa(t)$ is real continuous function that is independent of $\lambda$, doesn’t change the sign on the interval of integration $a \leq t \leq b$ (say it is positive), and has a single maximum on that interval.

Let’s consider simple examples of Laplace’s method.

**Example 1.** Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^\pi \frac{e^{\lambda \cos(x)}}{x^2 + 4} \, dx. \quad (3)$$

Since only small $|x|$, such that $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$, are important in the integral Eq. (11), we can approximate the integrand as following:

$$\cos(x) \approx 1 - \frac{x^2}{2}, \quad \rightarrow \quad e^{\lambda \cos(x)} \approx e^{\lambda} e^{-\frac{1}{2}x^2}, \quad (4)$$

$$\frac{1}{x^2 + 4} \approx \frac{1}{4}. \quad (5)$$

Thus,

$$I(\lambda) \sim \frac{1}{4} e^\lambda \int_0^\infty e^{-\frac{1}{2}x^2} \, dx = \frac{1}{8} \sqrt{\frac{2\pi}{\lambda}} e^\lambda = \sqrt{\frac{\pi}{32 \lambda}} e^\lambda. \quad (6)$$
Figure 2: Asymptotics Eq. (6) (solid line) compared to the numerically evaluated integral (3) (dashed line) for $2 \leq x \leq 8$. Notice the logarithmic scale on $y$ axis.

**Example 2.** Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^1 e^{-\lambda \sin^3(x)} \, dx.$$  

(7)

The maximum of the function in the exponent, $e^{-\sin^3 x}$ is at $x = 0$, so in this example the main contribution to the integral is coming from the vicinity of the left endpoint of the integration range, $x = 0$, where $\sin^3 x \sim x^3$.

$$I(\lambda) \sim \int_0^\infty e^{-\lambda x^3} \, dx.$$  

(8)

To evaluate the last integral, let’s introduce a new integration variable, $u = \lambda x^3$:

$$x^3 = \frac{u}{\lambda} \quad \rightarrow \quad x = \frac{u^{\frac{1}{3}}}{\lambda^{\frac{1}{3}}} \quad \rightarrow \quad dx = \frac{u^{\frac{1}{3}-1}}{3\lambda^{\frac{1}{3}}} \, du.$$  

(9)

$$I(\lambda) \sim \frac{1}{3\lambda^{\frac{1}{3}}} \int_0^\infty e^{-u} \frac{u^{\frac{1}{3}-1}}{3\lambda^{\frac{1}{3}}} \, du = \frac{\Gamma\left(\frac{1}{3}\right)}{3\lambda^{\frac{4}{3}}} = \frac{\Gamma\left(\frac{4}{3}\right)}{\lambda^{\frac{1}{3}}}.$$  

(10)

**Example 3.** Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_{-3}^{4} e^{-\lambda x^2} \log(1 + x^2) \, dx.$$  

(11)
Since only small \(|x|\), such that \(|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1\), are important in the integral Eq. (11) (for \(|x| \geq \frac{1}{\sqrt{\lambda}}\) the integrand is negligibly small due to the exponent’s factor), we can approximate the function in the integrand as following:

\[
\log\left(1 + x^2\right) \sim x^2.
\]  

Thus,

\[
I(\lambda) \sim 4 \int_{3}^{4} e^{-\lambda x^2} x^2 \, dx \sim \int_{-\infty}^{\infty} e^{-\lambda x^2} x^2 \, dx = 2 \int_{0}^{\infty} e^{-\lambda x^2} x^2 \, dx.
\]  

Introducing the new integration variable \(u\),

\[
u = \lambda x^2 \quad \rightarrow \quad x^2 = \frac{u}{\lambda} \quad \rightarrow \quad x = \frac{1}{\sqrt{\lambda}} u^{\frac{1}{2}} \quad \rightarrow \quad dx = \frac{1}{2\sqrt{\lambda}} u^{-\frac{1}{2}} \, du.
\]

\[
I(\lambda) \sim \lambda^{-\frac{3}{2}} \int_{0}^{\infty} e^{-u} u^{\frac{1}{2}} \, du = \lambda^{-\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) = \lambda^{-\frac{3}{2}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \lambda^{-\frac{3}{2}}
\]

\[
\text{Example 4.} \quad \text{Find the leading term of the asymptotics of the following integral for } n \gg 1:\n\]

\[
I(n) = \int_{-1}^{1} (\cos x)^n \, dx,
\]
Figure 4: Asymptotics Eq. (15) (solid line) compared to the numerically evaluated integral Eq. (11) (dashed line) for $10 \leq \lambda \leq 50$.

Since only small $|x|$, such that $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$, are important in the integral, we can approximate

$$\cos x \sim 1 - \frac{x^2}{2} \sim e^{-\frac{x^2}{2}}.$$  \hfill (17)

Thus,

$$I(n) = \int_{-1}^{1} \left( e^{-\frac{x^2}{2}} \right)^n dx \sim \int_{-\infty}^{\infty} e^{-\frac{nx^2}{2}} dx = \int_{-\infty}^{\infty} e^{-\left(\sqrt{n}x\right)^2} dx = \sqrt{\frac{2}{n}} \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\frac{2\pi}{n}}.$$ \hfill (18)

$$= \sqrt{\frac{2\pi}{n}}$$ \hfill (19)

Example 5. Find the leading term of the asymptotics of gamma function, $\Gamma(x)$, for $x \to \infty$:

$$\Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt = \int_{0}^{\infty} e^{-t + x\log t} \frac{1}{t} dt$$ \hfill (21)

The function in the exponent in Eq. (21),

$$f(t) = -t + x \log t,$$ \hfill (22)
Figure 5: Asymptotics Eq. (20) (solid line) compared to the numerically evaluated integral (16) (dashed line) for $10 \leq n \leq 50$.

has its maximum at $t = t_0$ which depends upon $x$:

$$\frac{df}{dt} = 0, \quad \rightarrow \quad -1 + \frac{x}{t} = 0 \quad \rightarrow \quad t_0 = x. \quad (23)$$

To make the maximum independent of $x$, let's introduce a new integration variable, $s$,

$$s = \frac{t}{x}, \quad \rightarrow \quad t = xs, \quad \rightarrow \quad dt = x\,ds, \quad \frac{dt}{t} = \frac{ds}{s}, \quad (24)$$

$$f(t) = -t + x \log t = -xs + x \log s + x \log x. \quad (25)$$

$$\Gamma(x) = e^{x \log x} \int_0^\infty e^{-x(s-\log s)} \frac{1}{s} \,ds. \quad (26)$$

Let's apply the Laplace’s method to the integral in Eq. (26):

$$f(s) = s - \log s, \quad \frac{df}{ds} = 1 - \frac{1}{s}. \quad (27)$$

$$\frac{df}{ds} = 0, \quad \rightarrow \quad s_0 = 1. \quad (28)$$

$$f(s_0) = 1, \quad \frac{d^2f}{ds^2} = \frac{1}{s^2}, \quad \rightarrow \quad \frac{d^2f}{ds^2}(s_0) = 1. \quad (29)$$

$$f(s) \approx f(s_0) + \frac{1}{2} \frac{d^2f}{ds^2}(s_0)(s-s_0)^2 = 1 + \frac{1}{2}(s-1)^2. \quad (30)$$

$$\int_0^\infty e^{-xf(s)} \frac{1}{s} \,ds \sim \int_0^\infty e^{-x(1+\frac{1}{2}(s-1)^2)} \frac{1}{s_0} \,ds \sim e^{-x} \int_{-\infty}^\infty e^{-\frac{1}{2}x(s-1)^2} \,ds = e^{-x} \int_{-\infty}^\infty e^{-\frac{1}{2}xs^2} \,ds. \quad (31)$$
The last integral is a Gaussian one:

\[ \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \, ds = \sqrt{\frac{2\pi}{x}}, \quad (32) \]

therefore

\[ \int_{0}^{\infty} e^{-x(s-\log s)} \frac{1}{s} \, ds \sim e^{-x} \sqrt{\frac{2\pi}{x}}. \quad (33) \]

Finally, combining Eq. (26) and Eq. (33)

\[ \Gamma(x) \sim e^{x \log x} e^{-x} \sqrt{\frac{2\pi}{x}} = \sqrt{\frac{2\pi}{x}} \left( \frac{x}{e} \right)^x. \quad (34) \]

The expression Eq. (34) is the leading term of the Stirling approximation for Gamma function.

Figure 6: Asymptotics Eq. (34) (solid line) compared to the numerically evaluated integral (21) (dashed line) for 2 \leq x \leq 8. Notice the logarithmic scale on y axis.

References