In this note we evaluate Gaussian integral,

$$I_g \equiv \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}, \quad (1)$$

and Fresnel integrals,

$$I_c \equiv \int_0^\infty \cos(x^2) \, dx = \frac{\sqrt{\pi}}{2\sqrt{2}} \quad (2)$$

and

$$I_s \equiv \int_0^\infty \sin(x^2) \, dx = \frac{\sqrt{\pi}}{2\sqrt{2}} \quad (3)$$

using differentiation under the integral sign.

The two parts of this note can be read independently.
Gaussian integral

Consider the following integral

\[ J(x) = \int_0^\infty \frac{e^{-x^2(1+y^2)}}{1+y^2} \, dy. \]  

(4)

We know its values for \( x = 0 \):

\[ J(0) = \int_0^\infty \frac{dy}{1+y^2} = \arctan(\infty) = \frac{\pi}{2}, \]

(5)

and for \( x = \infty \):

\[ J(\infty) = 0. \]

(6)

The derivative of \( J(x) \),

\[
\frac{dJ}{dx} = -2x \int_0^\infty e^{-x^2(1+y^2)} \, dy = -2e^{-x^2} \int_0^\infty e^{-(xy)^2} \, d(xy)
= -2e^{-x^2} \int_0^\infty e^{-u^2} \, du
= -2e^{-x^2} I_g. \]

(7)

Integrating Eq. (7) with respect to \( x \) from 0 to \( \infty \), we obtain:

\[ J(\infty) - J(0) = -2I_g \int_0^\infty e^{-x^2} \, dx = -2I_g^2, \]

(8)

or, using Eq. (5) and Eq. (6),

\[ I_g^2 = \frac{J(0)}{2} = \frac{\pi}{4}, \]

(9)

i.e.

\[ I_g = \frac{\sqrt{\pi}}{2}. \]

(10)
Fresnel’s integrals

Consider the following integrals:

\[
c(x) = \int_{0}^{\infty} \frac{\cos(x^2(1+y^2))}{1+y^2} \, dy, \tag{11}
\]

\[
s(x) = \int_{0}^{\infty} \frac{\sin(x^2(1+y^2))}{1+y^2} \, dy. \tag{12}
\]

We know their values for \(x = 0\):

\[
c(0) = \int_{0}^{\infty} \frac{dy}{1+y^2} = \arctan(\infty) = \frac{\pi}{2}, \tag{13}
\]

\[
s(0) = 0, \tag{14}
\]

and for \(x = \infty\):

\[
c(\infty) = s(\infty) = 0. \tag{15}
\]

The derivative of \(c(x)\),

\[
\frac{dc}{dx} = -2x \int_{0}^{\infty} \sin(x^2(1+y^2)) \, dy = -2x \int_{0}^{\infty} \sin(x^2 + xy^2) \, dy
\]

\[
= -2 \sin(x^2) \int_{0}^{\infty} \cos(xy) \, dy - 2 \cos(x^2) \int_{0}^{\infty} \sin(xy) \, dy
\]

\[
= -2 \sin(x^2) \int_{0}^{\infty} \cos(u^2) \, du - 2 \cos(x^2) \int_{0}^{\infty} \sin(u^2) \, du
\]

\[
= -2 \sin(x^2) I_c - 2 \cos(x^2) I_s. \tag{16}
\]

Similarly,

\[
\frac{ds}{dx} = 2 \cos(x^2) I_c - 2 \sin(x^2) I_s. \tag{17}
\]
Integrating Eqs. (16), (17) with respect to \( x \) from 0 to \( \infty \), we obtain:

\[
c(\infty) - c(0) = -2 I_c \int_{0}^{\infty} \sin(x^2) \, dx - 2 I_s \int_{0}^{\infty} \cos(x^2) \, dx = -4 I_c I_s, \quad (18)
\]

\[
s(\infty) - s(0) = 2 I_c \int_{0}^{\infty} \cos(x^2) \, dx - 2 I_s \int_{0}^{\infty} \sin(x^2) \, dx = 2 I_c^2 - 2 I_s^2. \quad (19)
\]

Using the boundary conditions Eqs. (13)-(15), \( c(\infty) - c(0) = -\frac{\pi}{2} \), \( s(\infty) - s(0) = 0 \), therefore

\[
I_c I_s = \frac{\pi}{8}, \quad I_c = I_s. \quad (20)
\]

Finally,

\[
I_c = I_s = \frac{\sqrt{\pi}}{2\sqrt{2}}. \quad (21)
\]

References