Show all your work and indicate your reasoning in order to receive the credit. Present your answers in *low-entropy* form. Write your name on the problems page and enclose it together with your solutions. Use **only** the methods we introduced in class.

Name: __________________________

Date: __________________________

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**The problem:**

This problem concerns the equation

\[
\frac{d^2x}{dt^2} + \epsilon \left(x^2 - x - 2\right) \frac{dx}{dt} + x = 0, \tag{1}
\]

where \(\epsilon\) is a large parameter, \(\epsilon \gg 1\). Follow the procedure described in the course notes on relaxation oscillations, *Singular perturbation theory*, to find the period, \(T(\epsilon)\), and “amplitude” of the limit cycle of this equation.

Assume that Eq. (1) is already written in dimensionless form.

Consult the numerical solution of Eq. (1) – see Fig 1.

1. (15 points) Rewrite Eq. (1) to use a small parameter \(\nu = \frac{1}{\epsilon}\). Scale the time variable \(t = \nu^a \tau\) and use the method of dominant balance to identify two solvable differential equations that describe the system’s “slow” and “fast” motion.

2. (10 points) Integrate the “slow” differential equation.

3. (10 points) Reduce the “fast” differential equations (which is a second order differential equation) to a first order differential equation. (There is no need to completely integrate the equation.) Note that the “new fast” equation must include one integration constant.

4. Proceed to the matching of the solutions for the slow and the fast regimes.
Figure 1: Numerical solution of Eq. (1) for $\varepsilon = 10$. 
(a) (5 points) Start by following the oscillator as it moves from its largest positive
displacement. Use the “slow” differential equation to identify the point of the
transition from the slow to the fast regimes.

(b) (5 points) Use the x coordinate of the transition to find the integration constant
in the “new fast” equation.

(c) (5 points) Factor the right hand side of the “new fast” equation and determine
x coordinate of the transition from the fast to the slow motion. You found the
largest in absolute value negative displacement of the oscillator!

(d) (5 points) You now start following the oscillator as it moves toward positive
displacements. Use the “slow” differential equation to identify the point of
transition from the slow to the fast regimes.

(e) (5 points) You now have the coordinates of the beginning and the end of the slow
motion for negative x. Use the solution of the “slow” differential equation to
determine the duration of the “slow negative” part.

(f) (5 points) Use the x coordinate of the transition to find another integration con-
stant in the “new fast” equation.

(g) (5 points) Factor the right hand side of the “new fast” equation and determine
x coordinate of the transition from the fast to the slow motion. You found the
largest positive displacement of the oscillator!

(h) (5 points) You now have the coordinates of the beginning and the end of the slow
motion for positive x. Use the solution of the “slow” differential equation to
determine the duration of the “slow positive” part.

Neglect the duration of the fast part of the limit cycle. The total duration of the
slow parts is approximately the period of the oscillations.

5. (10 points) Solve Eq. (1) numerically for a sufficiently large value of ε of your choice.
As the initial conditions, choose the maximal positive displacement in the limit cycle
and zero velocity. Plot your solution for several periods of oscillations. Estimate
the period of the limit cycle from your graph. Compare the result of the numerical
calculations and the theoretical analysis.

6. (15 points) Provide a clear self-contained description of the problem you solved and
the solution steps.