

DIFFERENTIATING UNDER THE INTEGRAL SIGN

SPRING SEMESTER 2017

http://www.phys.uconn.edu/~rozman/Courses/P2400_17S/



Last modified: May 21, 2017

1 Case of the integrand depending on a parameter

$$\frac{d}{dt} \left(\int_a^b f(x, t) dx \right) = \int_a^b \left(\frac{\partial}{\partial t} f(x, t) \right) dx \quad (1)$$

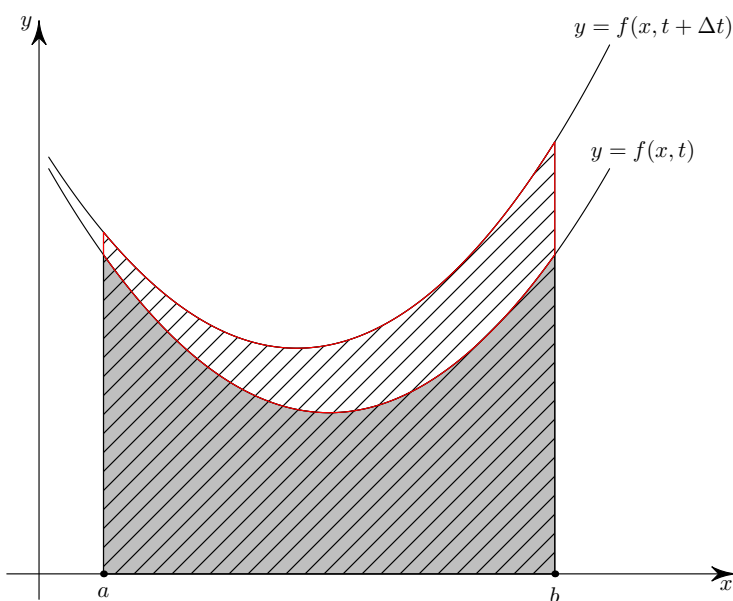
$$I(t) = \int_a^b f(x, t) dx \quad (2)$$

$$\frac{dI}{dt} = \lim_{\Delta t \rightarrow 0} \frac{I(t + \Delta t) - I(t)}{\Delta t} \quad (3)$$

$$I(t + \Delta t) - I(t) = \int_a^b [f(x, t + \Delta t) - f(x, t)] dx \quad (4)$$

$$= \int_a^b \left[\frac{\partial f}{\partial t} \Delta t + O(\Delta t^2) \right] dx = \left[\int_a^b \left(\frac{\partial f}{\partial t} \right) dx \right] \Delta t + O(\Delta t^2) \quad (5)$$

Figure 1: $I(t)$ (gray background), $I(t + \Delta t)$ (hatched background), and their difference in Eq. (4).



2 Case of the integration range depending on a parameter

$$\frac{d}{dt} \left(\int_{a(t)}^{b(t)} f(x) dx \right) = f(b(t)) \frac{db}{dt} - f(a(t)) \frac{da}{dt} \quad (6)$$

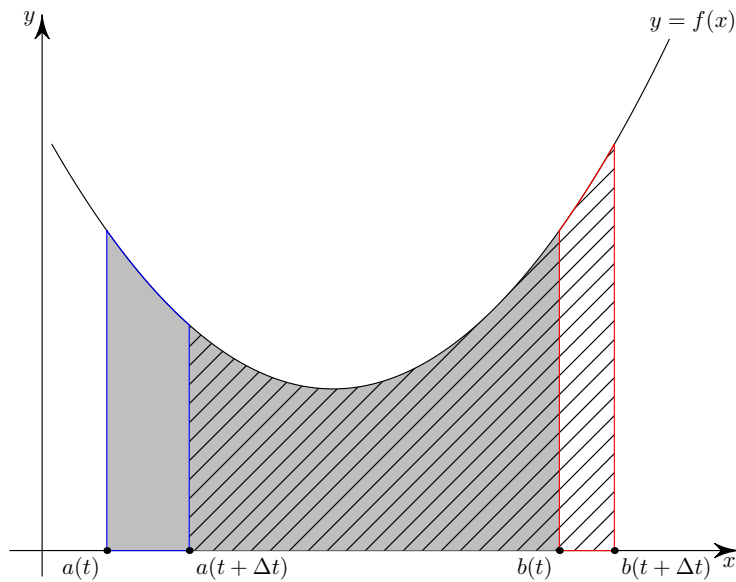
$$I(t) = \int_{a(t)}^{b(t)} f(x) dx \quad (7)$$

$$\frac{dI}{dt} = \lim_{\Delta t \rightarrow 0} \frac{I(t + \Delta t) - I(t)}{\Delta t} \quad (8)$$

$$I(t + \Delta t) - I(t) = (b(t + \Delta t) - b(t)) f(b(t)) - (a(t + \Delta t) - a(t)) f(a(t)) + O(\Delta t^2) \quad (9)$$

$$= \left[f(b(t)) \frac{db}{dt} - f(a(t)) \frac{da}{dt} \right] \Delta t + O(\Delta t^2) \quad (10)$$

Figure 2: $I(t)$ (gray background), $I(t + \Delta t)$ (hatched background), and their difference in Eq. (9).



3 General case

$$\frac{d}{dt} \left(\int_{a(t)}^{b(t)} f(x, t) dx \right) = \int_{a(t)}^{b(t)} \left(\frac{\partial}{\partial t} f(x, t) \right) dx + f(b(t)) \frac{db}{dt} - f(a(t)) \frac{da}{dt} \quad (11)$$

Figure 3: $I(t)$ (gray background), $I(t + \Delta t)$ (hatched background), and their difference in Eq. (9).

