INDUCED EMF IN A CIRCULAR LOOP

LECTURE NOTES, SPRING SEMESTER 2017

http://www.phys.uconn.edu/~rozman/Courses/P2400_17S/

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A circular wire loop of resistance $R$ and radius $a$ has its center at a distance $d_0 \ (d_0 > a)$ from a long straight wire. The wire is in the plane of the loop. (See Fig. 1.) The current in the long wire is changing, $I = I(t)$. What is the current in the loop?

Figure 1: A long straight wire carrying current $I(t)$ and a circular wire loop with the center distance $d_0$ from the wire. The wire is in the plane of the loop.

The magnitude of the emf induced in the loop is

$$\varepsilon = \left| \frac{d\Phi}{dt} \right| = \left| \frac{d}{dt} \int B_n \, dA \right|, \quad (1)$$
where $\Phi$ is the magnetic flux through the loop,

$$\Phi = \int B_n \, dA,$$

(2)

$B_n$ is the component of the magnetic field perpendicular to the plane of the loop; the integration is over the area of the loop. The magnetic field produced by the wire at a distance $d$ from the wire,

$$B_n = \frac{\mu_0 I}{2\pi d}.$$  

(3)

At a point inside the loop,

$$d = d_0 + r \cos \theta$$

(4)

and

$$dA = r \, dr \, d\theta,$$

(5)

so that the flux through the loop is

$$\Phi = \frac{\mu_0 I}{2\pi} \int_0^a \frac{r \, dr \, d\theta}{d_0 + r \cos \theta} = \frac{\mu_0 I}{2\pi} \int_0^a \phi(r) \, r \, dr,$$

(6)

where

$$\phi(r) \equiv \int_0^{2\pi} \frac{d\theta}{d_0 + r \cos \theta}.$$  

(7)

Change the integration variable:

$$z = e^{i\theta}, \quad d\theta = \frac{dz}{iz}, \quad \cos \theta = \frac{1}{2} \left(z + \frac{1}{z}\right).$$

(8)

In the $z$-plane the integration contour is the unit circle $|z| = 1$.

$$\phi(r) = -\frac{2i}{r} \oint_{|z|=1} \frac{dz}{z^2 + 2\frac{d_0}{r}z + 1}$$

(9)

The poles of the integrand are given by the roots of the quadratic polynomial in the denominator:

$$z^2 + 2\frac{d_0}{r}z + 1 = 0.$$  

(10)

$$z_{in, out} = -\frac{d_0}{r} \pm \sqrt{\left(\frac{d_0}{r}\right)^2 - 1}.$$  

(11)
Since the loop is not crossing the wire, $d_0 > a \geq r$, i.e. $\frac{d_0}{r} > 1$. Therefore both $z_{in}$ and $z_{out}$ are real. Since $z_{in}z_{out} = 1$, only one root, the one with the smaller absolute value, is inside the integration contour:

$$z_{in} = -\frac{d_0}{r} + \sqrt{\left(\frac{d_0}{r}\right)^2 - 1}. \quad (12)$$

![Figure 2: Integration contour for Eq. (9). The poles of the integrand, $z_{in}$ and $z_{out}$, are shown for $\frac{d_0}{r} = \frac{3}{2}$.](image)

$$\phi(r) = 2\pi i \left(-\frac{2i}{r}\right) \text{Res}\left(\frac{1}{(z-z_{in})(z-z_{out})}, z = z_{in}\right) = \frac{4\pi}{r} \frac{1}{z_{in} - z_{out}} = \frac{2\pi}{r} \frac{1}{\sqrt{\left(\frac{d_0}{r}\right)^2 - 1}} = \frac{2\pi}{\sqrt{d_0^2 - r^2}}. \quad (13)$$

$$\Phi = \frac{\mu_0 I}{2\pi} \int_0^a \phi(r) r \, dr = \frac{\mu_0 I}{2} \int_0^a \frac{r \, dr}{\sqrt{d_0^2 - r^2}}. \quad (15)$$

Noting that

$$r \, dr = \frac{1}{2} d(r^2) = -\frac{1}{2} d\left(d_0^2 - r^2\right) \quad (16)$$

and introducing new integration variable

$$u = d_0^2 - r^2, \quad d_0^2 - a^2 \leq u \leq d_0^2. \quad (17)$$
\[ \Phi = \frac{\mu_0 I}{2} \int_{d_0^2 - a^2}^{d_0^2} \frac{du}{\sqrt{u}} = \mu_0 I \sqrt{u} \bigg|_{u=d_0^2-a^2}^{u=d_0^2} = \mu_0 I \left( d_0 - \sqrt{d_0^2 - a^2} \right). \quad (18) \]

We can check the result by looking at the limit \( a \ll d_0 \). We expect the flux to be approximately

\[ \Phi \approx \frac{\mu_0 I}{2\pi d_0} \pi a^2 = \frac{\mu_0 I a^2}{2d_0}. \quad (19) \]

Now, if we expand the square root in Equation (18), we get

\[ \Phi = \mu_0 I d_0 \left( 1 - \sqrt{1 - \left( \frac{a}{d_0} \right)^2} \right) \approx \mu_0 I d_0 \left[ 1 - 1 + 1 + 1 \left( \frac{a^2}{d_0^2} \right) \right] = \frac{\mu_0 I a^2}{2d_0} \quad (20) \]

as expected.

Finally, the current in the loop is

\[ I = \frac{\varepsilon}{R} = \frac{\mu_0 I}{R} \left( d_0 - \sqrt{d_0^2 - a^2} \right), \quad (21) \]

where \( \dot{I} = \frac{dI}{dt} \).

**References**