Cauchy’s theorem states that if \( f(z) \) is analytic at all points on and inside a closed complex contour \( C \), then the integral of the function around that contour vanishes:

\[
\oint_C f(z) \, dz = 0.
\] (1)

1 A trigonometric integral

**Problem:** Show that

\[
\int_{\alpha \pi}^{\pi} \cos(\alpha \phi) [\cos \phi]^{\alpha-1} \, d\phi = 2^\alpha B(\alpha, \alpha) = 2^\alpha \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}.
\] (2)

**Solution:**

Recall the definition of Beta function,

\[
B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \, dx,
\] (3)
and consider the integral:

\[ J = \oint_C \left[ z(1 - z) \right]^{\alpha - 1} dz = 0, \quad \alpha > 1, \quad (4) \]

where the integration is over closed contour shown in Fig. 1. Since the integrand in Eq. (4) is analytic inside \( C \),

\[ J = 0. \quad (5) \]

On the other hand,

\[ J = J_I + J_{II}, \quad (6) \]

where \( J_I \) is the integral along the segment of the positive real axis, \( 0 \leq x \leq 1 \); \( J_{II} \) is the integral along the circular arc or radius \( R = \frac{1}{2} \) centered at \( z = \frac{1}{2} \).

Along the real axis \( z = x \), \( dz = dx \), thus

\[ J_I = \int_0^1 x^{\alpha-1} (1 - x)^{\alpha-1} dx = B(\alpha, \alpha). \quad (7) \]

Along the semi-circular arc

\[ z = \frac{1}{2} + \frac{1}{2} e^{i\theta} = \frac{1}{2} e^{i\theta} \left( e^{-i\theta/2} + e^{i\theta/2} \right) = e^{i\theta/2} \cos \frac{\theta}{2}, \quad (8) \]

where \( 0 < \theta < \pi \), and

\[ 1 - z = \frac{1}{2} - \frac{1}{2} e^{i\theta} = -i e^{i\theta/2} \sin \frac{\theta}{2}. \quad (9) \]

Figure 1: Integration contour for Problem 1

---

**Figure 1: Integration contour for Problem 1**
Hence,
\[ z(1 - z) = -i e^{i\theta} \cos \frac{\theta}{2} \sin \frac{\theta}{2} = -i e^{i\theta} \sin \theta. \]  
(10)
\[ dz = \frac{i}{2} e^{i\theta} d\theta. \]  
(11)

Therefore,
\[ J_{II} = -\left(\frac{-i}{2}\right)^\alpha \int_0^\pi e^{i\alpha \theta} (\sin \theta)^{\alpha-1} d\theta = -2^{-\alpha} e^{-i\pi \alpha} \int_0^\pi e^{i\alpha \theta} (\sin \theta)^{\alpha-1} d\theta, \]  
(12)
where we used that
\[ -i = e^{-i\frac{\pi}{2}}. \]  
(13)

Combining Eqs. (5), (6), (7), and (12) we obtain:
\[ 2^{-\alpha} e^{-i\frac{\pi \alpha}{2}} \int_0^\pi e^{i\alpha \theta} (\sin \theta)^{\alpha-1} d\theta = B(\alpha, \alpha). \]  
(14)

Transforming the expression on the left as following:
\[ \int_0^\pi e^{i\alpha (\theta - \frac{\pi}{2})} (\sin \theta)^{\alpha-1} d\theta = \int_0^\pi e^{i\alpha (\theta - \frac{\pi}{2})} \left[ \cos \left( \theta - \frac{\pi}{2} \right) \right]^{\alpha-1} d\theta \]  
(15)
\[ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\alpha \phi} \left[ \cos \phi \right]^{\alpha-1} d\phi \]  
(16)
\[ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\alpha \phi) \left[ \cos \phi \right]^{\alpha-1} d\phi \]  
(17)

finally obtain the relation:
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\alpha \phi) \left[ \cos \phi \right]^{\alpha-1} d\phi = 2^\alpha B(\alpha, \alpha) = 2^\alpha \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}. \]  
(18)
2 Euler’s log-sine integral.

Problem: Show that
\[ \int_0^\pi \log(\sin x) \, dx = -\pi \log 2. \] (19)

Integral Eq. (19), is called Euler’s log-sine integral. It was first evaluated (by Euler) in 1769.

Solution:
We start by integrating the function \( f(z) \),
\[ f(z) = \log \left( 1 - e^{2iz} \right), \] (20)
along the rectangular contour \( C \) with the corners at 0, \( \pi, \pi + iR, iR \), indented at the corners when necessary (see Fig. 2), and letting \( R \to \infty \).

\[ J = \oint_C \log \left( 1 - e^{2iz} \right) \, dz. \] (21)

On the one hand, the integrand in Eq. (21) is an analytic function inside \( C \), therefore
\[ J = 0. \] (22)

On the other hand,
\[ J = J_I + J_{II} + J_{III} + J_{IV} + J_V + J_{VI}, \] (23)
where the subscripts corresponds to integration contours labeled in Fig. 2.

Consider first the integrals \( J_{II} \) and \( J_{IV} \). The integrand \( f(z) \) is a periodic function with the period \( \pi \),
\[ f(z) = f(z + \pi). \] (24)

Indeed,
\[ f(z + \pi) = \log \left( 1 - e^{2i(z+\pi)} \right) = \log \left( 1 - e^{2iz} e^{2\pi i} \right) = \log \left( 1 - e^{2iz} \right) = f(z). \] (25)

Therefore in \( J_{II} \) and \( J_{IV} \) we have equal integrands but we are integrating in the opposite directions. Therefore,
\[ J_{II} = -J_{IV}, \] (26)
Figure 2: Integration contour for Problem 2
or

\[ J_{II} + J_{IV} = 0. \]  

(27)

Next, observe that \( f(z) \to 0 \) as \( y = \text{Im}(z) \to +\infty \):

\[ f(z) = \log\left(1 - e^{2i(x+iy)}\right) = \log\left(1 - e^{2ix} e^{-y}\right) \approx -e^{2ix} e^{-y} \to 0. \]  

(28)

Therefore,

\[ J_{III} = 0. \]  

(29)

Next, let’s show that

\[ J_{V} \equiv \lim_{r \to 0} \int_{C_V} \log\left(1 - e^{2iz}\right) \, dz = 0. \]  

(30)

Indeed, \( z = r e^{i\theta}, \, dz = i r e^{i\theta} \, d\theta, \, 0 \leq \theta \leq \frac{\pi}{2} \):

\[ J_{V} = i \lim_{r \to 0} r \int_{0}^{\frac{\pi}{2}} \log\left(1 - e^{2i e^{i\theta}}\right) e^{i\theta} \, d\theta \approx i \lim_{r \to 0} r \int_{0}^{\frac{\pi}{2}} \log\left(-2 i e^{i\theta}\right) e^{i\theta} \, d\theta = 0. \]  

(31)

Similarly we can show that

\[ J_{VI} = 0. \]  

(32)

Combining Eqs (22), (23), (27), (29), (30), and (32), we get that

\[ J_{I} = \int_{0}^{\pi} \log\left(1 - e^{2ix}\right) \, dx = 0. \]  

(33)

Rewriting the integrand in Eq. (33) as following,

\[
\log\left(1 - e^{2ix}\right) = \log\left[e^{ix} (e^{-ix} - e^{ix})\right] = \log\left[(e^{ix} + e^{-ix}) \frac{e^{ix} - e^{-ix}}{2i}\right] = \log\left(2 e^{i(x - \frac{\pi}{2})} \sin x\right) = \log 2 + i \left(x - \frac{\pi}{2}\right) + \log(\sin x),
\]

(34)

we obtain

\[
\int_{0}^{\pi} \left[\log 2 + i \left(x - \frac{\pi}{2}\right) + \log(\sin x)\right] \, dx = 0,
\]

(35)
or
\[ \int_0^\pi \log(\sin x) \, dx = -\pi \log 2. \] (36)

3 Another Euler integral

**Problem:** evaluate the following integral:

\[ I(\alpha) = \int_0^\infty \frac{\sin(x)}{x^\alpha} \, dx, \quad 0 < \alpha < 1. \] (37)

**Solution:**

Let’s consider the following integral:

\[ J(\alpha) = \oint_C \frac{e^{iz}}{z^\alpha} \, dz, \] (38)

where the integration contour \( C \) is sketch in Fig. 3.

The integrand in Eq. (38) is an analytic function inside \( C \), therefore

\[ J(\alpha) = 0. \] (39)

On the other hand,

\[ J(\alpha) = J_I + J_{II} + J_{III} + J_{IV}. \] (40)

where the subscripts corresponds to integration contours labeled in Fig. 3.

Let’s consider \( J_I, J_{II}, J_{III}, \) and \( J_{IV} \) separately:

\( J_I \): the integration is along the real axis, so \( z = x, \, dz = dx, \, r \leq x \leq R: \)

\[ J_I = \lim_{r \to 0} \lim_{R \to \infty} \int_r^R \frac{e^{ix}}{x^\alpha} \, dx = \int_0^\infty \frac{e^{ix}}{x^\alpha} \, dx, \] (41)

so

\[ I(\alpha) = \text{Im} J_I. \] (42)
$J_\Pi$: the integration is counterclockwise along the quarter-circle of radius $R$, $z = Re^{i\theta}$, $dz = iRe^{i\theta} \, d\theta$, $0 \leq \theta \leq \frac{\pi}{2}$:

$$J_\Pi = \lim_{R \to \infty} iR^{(1-\alpha)} \int_0^{\frac{\pi}{2}} e^{iR \cos \theta} e^{-R \sin \theta} e^{i(1-\alpha)\theta} \, d\theta.$$ (43)
For the absolute value of $J_{II}$ we have the following estimates:

$$
|J_{II}| = \lim_{R \to \infty} \left| R^{(1-\alpha)} \int_0^{\pi/2} e^{iR \cos \theta} e^{-R \sin \theta} e^{i(1-\alpha)\theta} \, d\theta \right| \quad (44)
$$

$$
\leq \lim_{R \to \infty} R^{(1-\alpha)} \int_0^{\pi/2} \left| e^{iR \cos \theta} e^{-R \sin \theta} e^{i(1-\alpha)\theta} \right| \, d\theta \quad (45)
$$

$$
= \lim_{R \to \infty} R^{(1-\alpha)} \int_0^{\pi/2} e^{-R \sin(\theta)} \, d\theta \leq \lim_{R \to \infty} R^{(1-\alpha)} \int_0^{\pi/2} e^{-\frac{2R}{\pi} \theta} \, d\theta \quad (46)
$$

$$
= \lim_{R \to \infty} R^{(1-\alpha)} \frac{\pi}{2R} \int_0^R e^{-u} \, du = \frac{\pi}{2} \lim_{R \to \infty} R^{-\alpha} \left( 1 - e^{-R} \right) = 0, \quad (47)
$$

where we used the inequalities

$$
\sin(\phi) \geq \frac{2}{\pi} \theta \quad \rightarrow \quad e^{-\sin(\theta)} \leq e^{-\frac{2}{\pi} \theta} \quad \rightarrow \quad e^{-R \sin(\theta)} \leq e^{-\frac{2R}{\pi} \theta}, \quad (48)
$$

that are valid within the integration range $0 \leq \theta \leq \frac{\pi}{2}$, and introduce a new integration variable $u = \frac{2R}{\pi} \theta$.

Thus,

$$
J_{II} = 0. \quad (49)
$$

$J_{III}$: the integration is along the imaginary axis, so $z = iy$, $dz = i \, dy$, $r \leq y \leq R$:

$$
J_1 = \lim_{r \to 0} \lim_{R \to \infty} i^{(1-\alpha)} \int_0^r \frac{e^{-y}}{y^\alpha} \, dy = -e^{i \frac{\pi}{2} (1-\alpha)} \int_0^\infty e^{-y} y^{-\alpha} \, dy = -e^{i \frac{\pi}{2} (1-\alpha)} \Gamma(1-\alpha). \quad (50)
$$

$J_{IV}$: the integration is clockwise along the quarter-circle of radius $r$, $z = re^{i\theta}$, $dz = i \, re^{i\theta} \, d\theta$, $0 \leq \theta \leq \frac{\pi}{2}$:

$$
J_{IV} = \lim_{r \to 0} i \, r^{(1-\alpha)} \int_0^{\pi/2} e^{i \theta} e^{i(1-\alpha)\theta} \, d\theta \approx -\lim_{r \to 0} i \, r^{(1-\alpha)} \int_0^{\pi/2} e^{i(1-\alpha)\theta} \, d\theta = 0. \quad (51)
$$
Combining Eqs. (39), (40), and (51), we get
\[ J_1 = e^{i \pi/2} (1 - \alpha) \Gamma(1 - \alpha). \] (52)

Taking the imaginary part, and using Eq. (42), we obtain
\[ \int_0^\infty \frac{\sin(x)}{x^\alpha} \, dx = \sin\left(\frac{\pi}{2} (1 - \alpha)\right) \Gamma(1 - \alpha). \] (53)

For the case \( \alpha = 1 \),
\[ \int_0^\infty \frac{\sin(x)}{x} \, dx = \lim_{\alpha \to 1} \frac{\pi}{2} (1 - \alpha) \Gamma(1 - \alpha) = \frac{\pi}{2} \Gamma(2 - \alpha) = \frac{\pi}{2} \Gamma(1) = \frac{\pi}{2}. \]

### 4 Fresnel integrals

**Problem:** Assuming that the value of the Gaussian integral is known,
\[ I = \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}, \] (54)
evaluate the Fresnel integrals,
\[ C = \int_0^\infty \cos(x^2) \, dx \] (55)
and
\[ S = \int_0^\infty \sin(x^2) \, dx. \] (56)

The integrals \( C \) and \( S \) are named after the Fresnel (French physicist, 1788-1827). They were first evaluated by Euler in 1781.

**Solution:**
Let’s pack \( C \) and \( S \) together:
\[ F = C + i S = \int_0^\infty \left[ \cos(x^2) + i \cos(x^2) \right] \, dx = \int_0^\infty e^{ix^2} \, dx, \] (57)
such that

\[ C = \text{Re} F \]  \hspace{1cm} (58)

and

\[ S = \text{Im} F. \]  \hspace{1cm} (59)

Consider the integral

\[ J = \int_C e^{iz^2} \, dz, \]  \hspace{1cm} (60)

where \( C \) is the contour in the complex plane shown in Fig. 4.

Since the integrand in Eq. (60) is analytic inside \( C \),

\[ J = 0. \]  \hspace{1cm} (61)

On the other hand,

\[ J = J_1 + J_{II} + J_{III}, \]  \hspace{1cm} (62)

where \( J_1 \) is the integral along the positive real axis, \( J_{II} \) is the integral along the circular arc or radius \( R \to \infty \), \( 0 \leq \theta \leq \frac{\pi}{4} \), and \( J_{III} \) is the integral from infinity to the origin along the ray that makes the angle \( \theta = \frac{\pi}{4} \) with the real axis.

Let’s consider \( J_1, J_{II}, \) and \( J_{III} \) separately:

\( J_1 \): the integration is along the real axis, so \( z = x \), \( dz = dx \), \( 0 \leq x \leq \infty \):

\[ J_1 = \int_0^{\infty} e^{ix^2} \, dx. \]
\[ J_1 = \int_{C_1} e^{iz^2} \, dz = \int_{0}^{\infty} e^{ix^2} \, dx = F. \] (63)

\[ J_{II}: \text{the integration is along the circular arc of radius } R \text{ so } z = Re^{i\theta}, \text{ } dz = iRe^{i\theta} \, d\theta, \]
\[ z^2 = R^2 \, e^{2i\theta} = R^2 \left( \cos(2\theta) + i \sin(2\theta) \right), \quad 0 \leq \theta \leq \frac{\pi}{4}; \]
\[ J_{II} = \int_{C_{II}} e^{iz^2} \, dz = iR \int_{0}^{\frac{\pi}{4}} e^{iR^2 \cos(2\theta)} e^{-R^2 \sin(2\theta)} \, d\theta. \] (64)

For the absolute value of \( J_{II} \) we have the following estimates:
\[
|J_{II}| = \left| R \int_{0}^{\frac{\pi}{4}} e^{iR^2 \cos(2\theta)} e^{-R^2 \sin(2\theta)} \, d\theta \right| \leq R \int_{0}^{\frac{\pi}{4}} e^{iR^2 \cos(2\theta)} e^{-R^2 \sin(2\theta)} \, d\theta \]
\[ = R \int_{0}^{\frac{\pi}{4}} e^{-R^2 \sin(2\theta)} \, d\theta = \frac{R}{2} \int_{0}^{\frac{\pi}{4}} e^{-R^2 \sin(\phi)} \, d\phi < \frac{R}{2} \int_{0}^{\frac{\pi}{4}} e^{-\frac{2R^2}{\pi} \phi} \, d\phi \] (65)
\[ = \frac{R}{2} \frac{\pi}{2R^2} \int_{0}^{R^2} e^{-u} \, du = \frac{\pi}{4R} \left( 1 - e^{-R^2} \right) < \frac{\pi}{4R}, \] (66)

where we introduced a new integration variable \( \phi = 2\theta \), used the inequalities
\[
\sin(\phi) \geq \frac{2}{\pi} \phi \quad \rightarrow \quad e^{-\sin(\phi)} \leq e^{-\frac{2}{\pi} \phi} \quad \rightarrow \quad e^{-R^2 \sin(\phi)} \leq e^{-\frac{2R^2}{\pi} \phi}, \] (68)

that are valid within the integration range \( 0 \leq \phi \leq \frac{\pi}{4} \), and introduce a new integration variable \( u = \frac{2R^2}{\pi} \phi \).

Thus we obtained that
\[ |J_{II}| < \frac{\pi}{4R}. \] (69)

Therefore,
\[ J_{II} = 0 \] (70)
as \( R \to \infty \).
$J_{III}$: the integration is along the ray making the angle $\frac{\pi}{4}$ with the real axis so $z = re^{i\frac{\pi}{4}}$, $z^2 = r^2 e^{i\frac{\pi}{2}} = ir^2$, $dz = e^{i\frac{\pi}{4}} dr$, $0 \leq r < \infty$.

\[
J_{III} = \int_{C_{III}} e^{iz^2} \, dz = e^{i\frac{\pi}{4}} \int_{0}^{\infty} e^{-r^2} \, dr = -e^{i\frac{\pi}{4}} \int_{0}^{\infty} e^{-r^2} \, dr = -e^{i\frac{\pi}{4}} \frac{\sqrt{\pi}}{2}. \tag{71}
\]

Combining Eqs. (61), (63), (70), and (71) we obtain:

\[
F = \frac{\sqrt{\pi}}{2} e^{-\frac{\pi}{4}}. \tag{72}
\]

Finally, the Fresnel’s integrals are:

\[
C = \text{Re} \, F = \frac{\sqrt{\pi}}{2} \cos \left( \frac{\pi}{4} \right) = \sqrt{\frac{\pi}{8}} \tag{73}
\]

and

\[
S = -\text{Im} \, F = \frac{\sqrt{\pi}}{2} \sin \left( \frac{\pi}{4} \right) = \sqrt{\frac{\pi}{8}}. \tag{74}
\]