Mathematical methods for the physical sciences
Things to know for Midterm I

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1. Gamma function, $\Gamma(x)$:

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, \mathrm{d}x$$

$$\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}, \quad \Gamma(1) = 1$$

$$\Gamma(x) = (x-1)\,\Gamma(x-1)$$

$$\Gamma(n) = (n-1)!$$

2. Beta function, $B(x,y)$:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{d}t$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

3. Frulliani's integral:

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\, \mathrm{d}x = [f(\infty) - f(0)]\, \ln\frac{a}{b}$$

4. Leibniz's formula:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(t,x)\, \mathrm{d}t = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(t,x)\, \mathrm{d}t + f(b(x),x)\frac{\mathrm{d}b}{\mathrm{d}x} - f(a(x),x)\frac{\mathrm{d}a}{\mathrm{d}x}$$

5. Euler's formula:

$$e^{ix} = \cos(x) + i\sin(x), \quad \text{where} \quad i \equiv \sqrt{-1}.$$

In particular,

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right),$$
$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right),$$
$$e^{\pm i\pi} = -1, \quad e^{\pm i\frac{\pi}{2}} = \pm i.$$

6. Complex numbers – coordinate and polar form:

$$z = x + iy = re^{i\varphi}, \quad \text{where} \quad i \equiv \sqrt{-1} = e^{i\frac{\pi}{2}}.$$

$$r \equiv |z| = \sqrt{x^2 + y^2}, \quad \tan\varphi = \frac{y}{x} \qquad \longleftrightarrow \qquad x = r\cos\varphi, \quad y = r\sin\varphi.$$

7. Cauchy-Riemann equations:

$$f(z) = u(x, y) + iv(x, y).$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}. \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

8. Cauchy's theorem:

$$\oint_C f(z)\mathrm{d}z = 0,$$

if $f(z)$ is analytic inside an arbitrary closed contour C. Alternatively,

$$\int_{C_1} f(z)\mathrm{d}z = \int_{C_2} f(z)\mathrm{d}z,$$

where C_1 and C_2 are two different complex paths, such that $C_1 - C_2 = C$, connecting a pair of points, z_1 and z_2 in the complex plane.

9. Cauchy's formula

$$\oint_C \frac{f(z)}{z - z_0}\mathrm{d}z = 2\pi i f(z_0),$$

if $f(z)$ is analytic inside a closed contour C.

10. Taylor series:

    exist if $f(z)$ is analytic in the vicinity of z_0.

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \ldots = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

$$a_n = \frac{1}{n!}\frac{\mathrm{d}^n f}{\mathrm{d}z^n}\bigg|_{z=z_0}.$$

11. Familiar series:

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \ldots + \frac{1}{n!}z^n + \ldots = \sum_{n=0}^{\infty}\frac{z^n}{n!}$$

$$\cos z = 1 - \frac{1}{2}z^2 + \frac{1}{4!}z^4 + \ldots = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$$

$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \ldots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!}$$

12. Laurent series:

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \ldots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \ldots,$$

$$f(z) = \sum_{n=-m}^{\infty} a_n(z - z_0)^n.$$

$f(z)$ has an isolated singularity at z_0; m - the order of the pole.

$$a_n = \frac{1}{(m+n)!}\frac{\mathrm{d}^{m+n}}{\mathrm{d}z^{m+n}}\left[(z - z_0)^m f(z)\right]\bigg|_{z=z_0}.$$

13. Residues:

$$\oint_C f(z)\mathrm{d}z = 2\pi i a_{-1},$$

where C is a closed contour that includes z_0, the single singularity of $f(z)$.

$$\mathrm{Res}(f(z), z = z_0) \equiv a_{-1}.$$

For a simple pole,

$$\mathrm{Res}(f(z), z = z_0) = \lim_{z \to z_0} \left[(z - z_0) f(z)\right].$$

If $f(z) = \frac{p(z)}{q(z)}$ then $q(z_0) = 0$ and

$$\mathrm{Res}(f(z), z = z_0) = \frac{p(z_0)}{q'(z_0)},$$

where $q' = \frac{\mathrm{d}q}{\mathrm{d}z}$.

For a pole of order m,

$$\mathrm{Res}(f(z), z = z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left[(z - z_0)^m f(z)\right].$$

14. Jordan's lemma:

    yields a simple way to calculate the integral along the real axis of functions $f(z) = e^{iaz} g(z)$.