Method of steepest descents

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The method of steepest descents is a technique for finding the asymptotic behavior of integrals of the form

$$I(\lambda) = \int_C h(t)\, e^{\lambda\rho(t)}\, \mathrm{d}t \tag{1}$$

as $\lambda \to \infty$, where C is an integration contour in the complex-t plane and $h(t)$ and $\rho(t)$ are analytic functions of t. The idea of the method is to use the analyticity of the integrand to justify deforming the contour C to a new contour C' on which $\rho(t)$ has a constant imaginary part. Once this has been done, $I(\lambda)$ may be evaluated asymptotically as $\lambda \to \infty$ using Laplace's method. Indeed, on the contour C we may write $\rho(t) = \phi(t) + i\psi$, where ψ is a real constant and $\phi(t)$ is a real function. Thus, $I(\lambda)$ takes the form

$$I(\lambda) = e^{i\lambda\psi} \int_C h(t)\, e^{\lambda\phi(t)}\, \mathrm{d}t. \tag{2}$$

Although t is complex, Eq. (2) can be treated by Laplace's method as $\lambda \to \infty$ because $\phi(t)$ is real.

Example 1. Find the leading terms of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^1 \cos(\lambda x)\, \log x\, \mathrm{d}x. \tag{3}$$

Let's consider the integral

$$J(\lambda) = \int_0^1 e^{i\lambda z}\, \log z\, \mathrm{d}z. \tag{4}$$

$$I(\lambda) = \mathrm{Re}\, J(\lambda). \tag{5}$$

To approximate $J(\lambda)$ we deform the integration contour OP, which runs from 0 to 1 along the real x axis, to one which consists of three line segments: OQ, which runs up the imaginary y axis from 0 to iR; QS, which runs parallel to the real x axis from iR to $1 + iR$; and SP, which runs down from $1 + iR$ to 1 along a straight line parallel to the imaginary y axis (see Fig. 1). By Cauchy's theorem,

$$J(\lambda) = \int_{OQSP} e^{i\lambda z}\, \log z\, \mathrm{d}z. \tag{6}$$

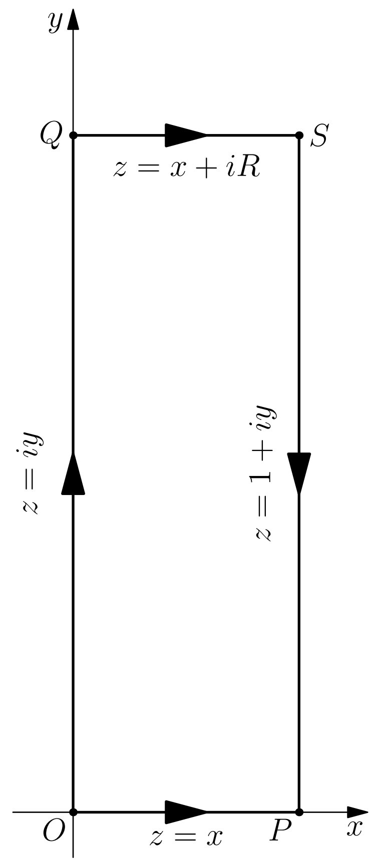


Figure 1: By Cauchy's theorem, the integral of analytic function over the contour OP is equal to the integral over the contour OQSP.

Next we let $R \to \infty$. In this limit the contribution from QS approaches 0 due to the exponential factor $e^{-\lambda R}$ in the integrand. In the integral along OQ we set $z = is$, and in the integral along SP we set $z = 1 + is$, where s is real in both integrals. This gives

$$I(\lambda) = \mathrm{Re}\int_0^\infty e^{-\lambda s}\,\log(is)\,\mathrm{d}(is) - \mathrm{Re}\int_0^\infty e^{-\lambda s + i\lambda}\,\log(1+is)\,\mathrm{d}(is) \tag{7}$$

The sign of the second integral on the right is negative because SP is traversed downward.

In the first integral in Eq. (7) which we evaluate exactly,

$$\log(is) \equiv \log\left(s\,e^{i\frac{\pi}{2}}\right) = \log s + i\frac{\pi}{2}, \tag{8}$$

thus

$$\mathrm{Re}\int_0^\infty e^{-\lambda s}\,\log(is)\,\mathrm{d}(is) = \int_0^\infty e^{-\lambda s}\,\mathrm{Re}\left(i\log s - \frac{\pi}{2}\right)\,\mathrm{d}s = -\frac{\pi}{2}\int_0^\infty e^{-\lambda s}\,\mathrm{d}s = -\frac{\pi}{2\lambda}. \tag{9}$$

In the second integral in Eq. (7),

$$\mathrm{Re}\int_0^\infty e^{-\lambda s + i\lambda}\,\log(1+is)\,\mathrm{d}(is) = \mathrm{Re}\,e^{i\lambda}\int_0^\infty e^{-\lambda s}\,\left[i\log(1+is)\right]\,\mathrm{d}s, \tag{10}$$

only the small values of s contribute to the integral due to the factor $e^{-\lambda s}$ in the integrand. Therefore,

$$\log(1+is) \approx is. \tag{11}$$

$$\begin{aligned}\operatorname{Re} e^{i\lambda}\int_0^\infty e^{-\lambda s}\left[i\log(1+is)\right]\mathrm{d}s \sim -\operatorname{Re}\left(e^{i\lambda}\right)\int_0^\infty e^{-\lambda s} s\,\mathrm{d}s &= -\frac{\cos\lambda}{\lambda^2}\int_0^\infty e^{-\lambda s}(\lambda s)\,\mathrm{d}(\lambda s)x \\ &= -\frac{\cos\lambda}{\lambda^2}\int_0^\infty e^{-u}\,u\,\mathrm{d}u \\ &= -\frac{\cos\lambda}{\lambda^2}. \end{aligned} \tag{12}$$

Combining Eq. (9) and Eq. (12),

$$I(\lambda) \sim \boxed{-\frac{\pi}{2\lambda}+\frac{\cos\lambda}{\lambda^2}}. \tag{13}$$

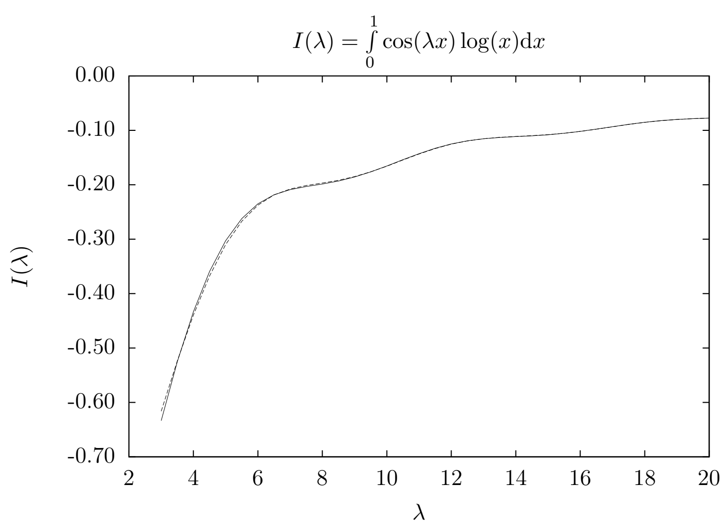


Figure 2: Asymptotics Eq. (13) (solid line) compared to numerically evaluated Eq. (3) (dashed line) for $2 \le \lambda \le 20$.

Example 2. Find the leading terms of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^1 \cos\left(\lambda\left(x+\frac{1}{2}\right)^2\right)\frac{1}{\sqrt{x}}\,\mathrm{d}x. \tag{14}$$

Let's consider the integral

$$J(\lambda) = \int_0^1 e^{i\lambda\left(z+\frac{1}{2}\right)^2}\frac{1}{\sqrt{z}}\,\mathrm{d}z, \tag{15}$$

$$I(\lambda) = \text{Re}\, J(\lambda). \tag{16}$$

To approximate $J(\lambda)$ we deform the integration contour OP, which runs from 0 to 1 along the real x axis, to one which consists of three segments (see Fig. 3): (a) a steepest-descent contour C_1 passing through $z = 0$; (b) a steepest-descent contour C_3 passing through $z = 1$; (c) a contour C_2 which runs parallel to the real x axis at $y = R$ connecting C_1 and C_3.

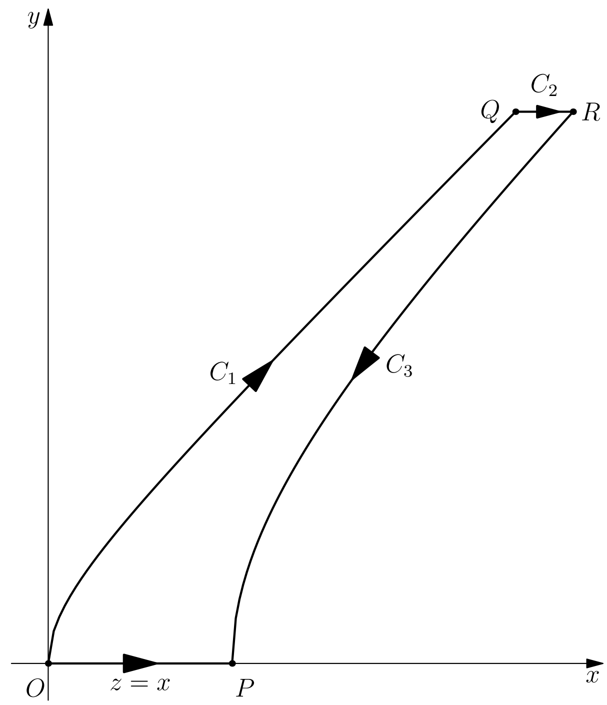


Figure 3: By Cauchy's theorem, the integral of analytic function over the contour OP is equal to the integral over the contour OQRP.

We let $R \to \infty$. In this limit the contribution from C_2 approaches 0 due to the exponential factor $e^{-\lambda R^2}$ in the integrand.

$$\rho(z) = i\left(z + \frac{1}{2}\right)^2 = i\left((x+iy) + \frac{1}{2}\right)^2 = i\left((x+\frac{1}{2})^2 - y^2\right) - 2\,y\,(x+\frac{1}{2}) \tag{17}$$

Next, we find a steepest-descent contour C_1 passing through point O, $z = 0$, along which $\text{Im}\rho(z)$ is constant. At $z = 0$, the value of $\text{Im}\rho(z)$ is $\frac{1}{4}$. Therefore, the constant-phase contour passing through $x = 0$, $y = 0$ is given by

$$\left(x + \frac{1}{2}\right)^2 - y^2 = \frac{1}{4}, \tag{18}$$

or

$$x = \sqrt{y^2 + \frac{1}{4}} - \frac{1}{2}, \quad 0 \le y < \infty. \tag{19}$$

To evaluate the contribution to $I(\lambda)$ from the integral on C_1 let's change the integration variable from z to s where s is defined by

$$s = -\text{Re}\,\rho(z) = 2\,y\,(x + \frac{1}{2}) = 2\,y\sqrt{y^2 + \frac{1}{4}}. \tag{20}$$

Observe that s is real and satisfies $0 \leq s < \infty$ along C_1.

$$\rho(z) = i\left(z+\frac{1}{2}\right)^2 = \frac{i}{4} - s, \tag{21}$$

$$z = \sqrt{\frac{1}{4}+is} - \frac{1}{2}, \tag{22}$$

$$\mathrm{d}z = \frac{i}{2}\frac{\mathrm{d}s}{\sqrt{\frac{1}{4}+is}}. \tag{23}$$

$$J_1 = \frac{ie^{\frac{i\lambda}{4}}}{2}\int_0^\infty \frac{e^{-\lambda s}\mathrm{d}s}{\sqrt{\sqrt{\frac{1}{4}+is}-\frac{1}{2}}\sqrt{\frac{1}{4}+is}}. \tag{24}$$

Equation (24) is a Laplace type integral, therefore for $\lambda \to \infty$ the main contribution comes from the small values of s.

$$\sqrt{\frac{1}{4}+is} \approx \frac{1}{2}, \tag{25}$$

$$\sqrt{\sqrt{\frac{1}{4}+is}-\frac{1}{2}} = \sqrt{\frac{1}{2}\left(\sqrt{1+4is}-1\right)} \approx \sqrt{is} = s^{\frac{1}{2}}e^{i\frac{\pi}{4}}. \tag{26}$$

$$\begin{aligned} J_1 \sim e^{\frac{i\lambda}{4}}e^{i\frac{\pi}{4}}\int_0^\infty e^{-\lambda s}s^{-\frac{1}{2}}\mathrm{d}s &= \frac{1}{\sqrt{\lambda}}e^{i\left(\frac{\lambda}{4}+\frac{\pi}{4}\right)}\int_0^\infty e^{-\lambda s}(\lambda s)^{-\frac{1}{2}}\mathrm{d}(\lambda s) \\ &= \frac{1}{\sqrt{\lambda}}e^{i\left(\frac{\lambda}{4}+\frac{\pi}{4}\right)}\int_0^\infty e^{-u}u^{-\frac{1}{2}}\mathrm{d} = \frac{1}{\sqrt{\lambda}}e^{i\left(\frac{\lambda}{4}+\frac{\pi}{4}\right)}\Gamma\left(\frac{1}{2}\right) \\ &= \sqrt{\frac{\pi}{\lambda}}e^{i\left(\frac{\lambda}{4}+\frac{\pi}{4}\right)}. \end{aligned} \tag{27}$$

Next, we find a steepest-descent contour C_3 passing through point P, $z=1$, along which $\mathrm{Im}\rho(z)$ is constant. At $z=1$, the value of $\mathrm{Im}\rho(z)$ is $\frac{9}{4}$. Therefore, the constant-phase contour passing through $x=1$, $y=0$ is given by

$$\left(x+\frac{1}{2}\right)^2 - y^2 = \frac{9}{4}, \tag{28}$$

or

$$x = \sqrt{y^2+\frac{9}{4}} - \frac{1}{2}, \quad 0 \leq y < \infty. \tag{29}$$

To evaluate the contribution to $I(\lambda)$ from the integral on C_3 let's change the integration variable from z to s where s is defined by

$$s = -\mathrm{Re}\,\rho(z) = 2\,y\,(x+\frac{1}{2}) = 2\,y\sqrt{y^2+\frac{9}{4}}. \tag{30}$$

Observe that s is real and satisfies $0 \le s < \infty$ along C_1.

$$\rho(z) = i\left(z+\frac{1}{2}\right)^2 = \frac{9i}{4} - s, \tag{31}$$

$$z = \sqrt{\frac{9}{4}+is} - \frac{1}{2}, \tag{32}$$

$$\mathrm{d}z = \frac{i}{2}\frac{\mathrm{d}s}{\sqrt{\frac{9}{4}+is}}. \tag{33}$$

$$J_3 = -\frac{ie^{\frac{9i\lambda}{4}}}{2}\int_0^\infty \frac{e^{-\lambda s}\mathrm{d}s}{\sqrt{\sqrt{\frac{9}{4}+is}-\frac{1}{2}}\sqrt{\frac{9}{4}+is}}. \tag{34}$$

The sign of the second integral on the right is negative because C_3 is traversed downward.

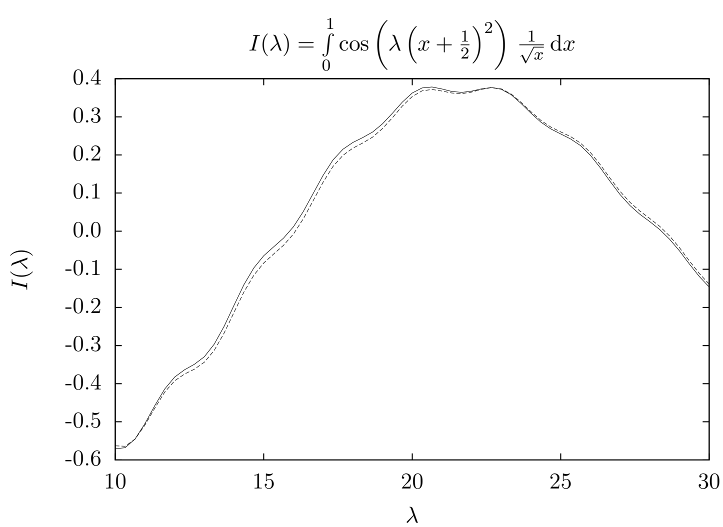
Equation (34) is a Laplace type integral, therefore for $\lambda \to \infty$ the main contribution comes from the small values of s.

$$\sqrt{\frac{9}{4}+is} \approx \frac{3}{2}, \tag{35}$$

$$\sqrt{\sqrt{\frac{9}{4}+is}-\frac{1}{2}} \approx 1. \tag{36}$$

$$\begin{aligned} J_3 \sim -\frac{1}{3}e^{\frac{9i\lambda}{4}}e^{i\frac{\pi}{2}}\int_0^\infty e^{-\lambda s}\mathrm{d}s &= -\frac{1}{3\lambda}e^{i\left(\frac{9\lambda}{4}+\frac{\pi}{2}\right)}\int_0^\infty e^{-\lambda s}\mathrm{d}(\lambda s) \\ &= -\frac{1}{3\lambda}e^{i\left(\frac{9\lambda}{4}+\frac{\pi}{2}\right)}. \end{aligned} \tag{37}$$

Figure 4: Asymptotics Eq. (38) (solid line) compared to numerically evaluated Eq. (14) (dashed line) for $10 \le \lambda \le 30$.



Combining Eq. (27) and Eq. (37),

$$I(\lambda) = \mathrm{Re}\left(\sqrt{\frac{\pi}{\lambda}} e^{i\left(\frac{\lambda}{4}+\frac{\pi}{4}\right)} - \frac{1}{3\lambda} e^{i\left(\frac{9\lambda}{4}+\frac{\pi}{2}\right)}\right) = \boxed{\sqrt{\frac{\pi}{\lambda}} \cos\left(\frac{\lambda}{4}+\frac{\pi}{4}\right) - \frac{1}{3\lambda}\cos\left(\frac{9\lambda}{4}+\frac{\pi}{2}\right)}. \tag{38}$$

References

[1] Lorella M. Jones. *An introduction to mathematical methods of physics*. Benjamin Cummings, 1979.

[2] Carl M. Bender and Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. Springer Verlag, 1999.