Method of stationary phase

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There is an immediate generalization of the Laplace integrals

$$\int_a^b f(t)e^{x\phi(t)}\mathrm{d}t \tag{1}$$

which we obtain by allowing the function $\phi(t)$ in Eq. (1) to be complex. We may assume that $f(t)$ is real; if it were complex, $f(t)$ could be decomposed into a sum of its real and imaginary parts. However, allowing $\phi(t)$ to be complex poses nontrivial problems. We consider the special case in which $\phi(t)$ is pure imaginary: $\phi(t) = i\psi(t)$ where $\psi(t)$ is real. The resulting integral

$$I(x) = \int_a^b f(t)e^{ix\psi(t)}\mathrm{d}t \tag{2}$$

with $f(t)$, $\psi(t)$, a, b, x all real is called a generalized Fourier integral. When $\psi(t) = t$, $I(x)$ is an ordinary Fourier integral.

The method of stationary phase gives the leading asymptotic behavior of generalized Fourier integrals having stationary points, $\psi' = 0$. This method is similar to Laplace's method in that the leading contribution to $I(x)$ comes from a small interval surrounding the stationary points of ψ.

Example 1. Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_{-3}^{4} \cos\left(\lambda \sinh^2(x)\right)\sqrt{1+x^2}\mathrm{d}x. \tag{3}$$

Since only small $|x|$, such that $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$ are important,

$$\sinh x \sim x, \tag{4}$$

$$\cos\left(\lambda \sinh^2(x)\right) \sim \cos\left(\lambda x^2\right) = \mathrm{Re}\, e^{i\lambda x^2} \tag{5}$$

$$\sqrt{1+x^2} \sim 1. \tag{6}$$

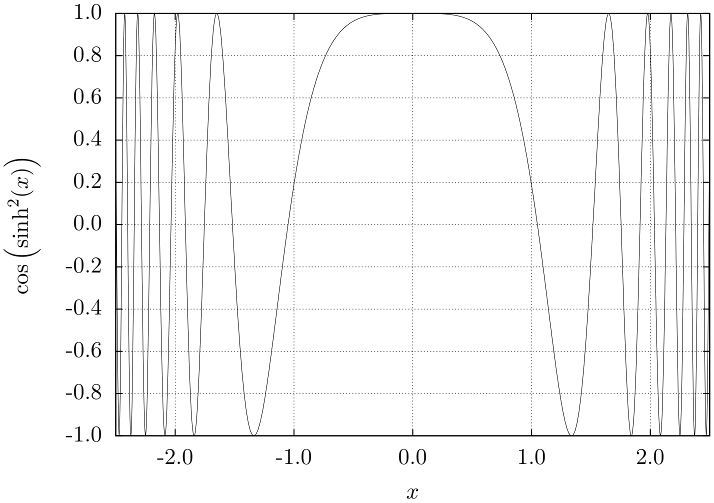

Figure 1: The graph of the oscillating factor, $\cos\left(\lambda \sinh^2(x)\right)$ in Eq. (3), for $\lambda = 1$.

$$I(\lambda) \sim \mathrm{Re} \int_{-3}^{4} e^{i\lambda x^2} \mathrm{d}x \sim \mathrm{Re} \int_{-\infty}^{\infty} e^{i\lambda x^2} \mathrm{d}x. \tag{7}$$

New integration variable,

$$u^2 = \lambda x^2 \quad \longrightarrow \quad x^2 = \frac{u^2}{\lambda} \quad \longrightarrow \quad x = \frac{u}{\sqrt{\lambda}} \quad \longrightarrow \quad \mathrm{d}x = \frac{1}{\sqrt{\lambda}} \mathrm{d}u. \tag{8}$$

$$I(\lambda) \sim \mathrm{Re} \frac{1}{\sqrt{\lambda}} \underbrace{\int_{-\infty}^{\infty} e^{iu^2} \, \mathrm{d}u}_{\sqrt{\pi} e^{i\frac{\pi}{4}}} = \sqrt{\frac{\pi}{\lambda}} \underbrace{\mathrm{Re}\left(e^{i\frac{\pi}{4}}\right)}_{\frac{1}{\sqrt{2}}} = \boxed{\sqrt{\frac{\pi}{2\lambda}}} \tag{9}$$

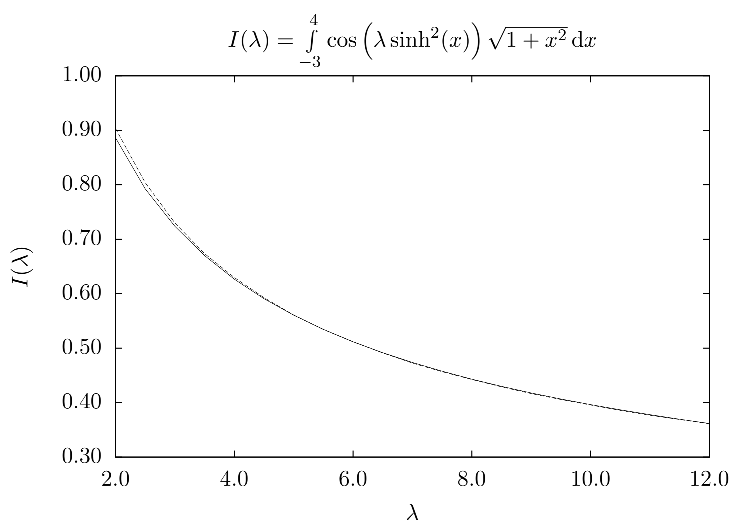

Figure 2: Asymptotics Eq. (9) (solid line) compared to numerically evaluated Eq. (3) (dashed line) for $2 \leq \lambda \leq 12$.

Example 2. Find the leading term of the asymptotics of the Bessel function $J_0(x)$ for $x \to \infty$:

$$J_0(x) = \frac{1}{\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(x \cos\theta\right) \mathrm{d}\theta \tag{10}$$

Bessel function $J_0(x)$ is a solution of the following second order linear differential equation:

$$xy'' + y' + xy = 0. \tag{11}$$

Let's show first that Eq. (10) is indeed a solution of Eq. (11).

$$\frac{\mathrm{d}}{\mathrm{d}x} J_0(x) = -\frac{1}{\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\left(x\cos\theta\right) \cos\theta \, \mathrm{d}\theta, \tag{12}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} J_0(x) = -\frac{1}{\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(x\cos\theta\right) \cos^2\theta \, \mathrm{d}\theta. \tag{13}$$

$$\begin{aligned}
x\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} J_0(x) + J_0(x)\right) &= \frac{x}{\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - \cos^2\theta\right) \cos\left(x\cos\theta\right) \, \mathrm{d}\theta & (14)\\
&= \frac{x}{\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \, \cos\left(x\cos\theta\right) \, \mathrm{d}\theta & (15)\\
&= -\frac{1}{\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta \, \cos\left(x\cos\theta\right) \, \mathrm{d}(x\cos\theta) & (16)\\
&= -\frac{1}{\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta \, \mathrm{d}\left(\sin\left(x\cos\theta\right)\right) & (17)\\
&= -\frac{1}{\pi} \sin\theta \, \sin\left(x\cos\theta\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\left(x\cos\theta\right) \cos\theta \, \mathrm{d}\theta & (18)\\
&= -\frac{\mathrm{d}}{\mathrm{d}x} J_0(x), & (19)
\end{aligned}$$

which is indeed in agreement with Eq. (11).

Let's rewrite integral Eq. (10) in the exponential form:

$$J_0(x) = \frac{1}{\pi} \mathrm{Re} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ix\cos\theta} \mathrm{d}\theta. \tag{20}$$

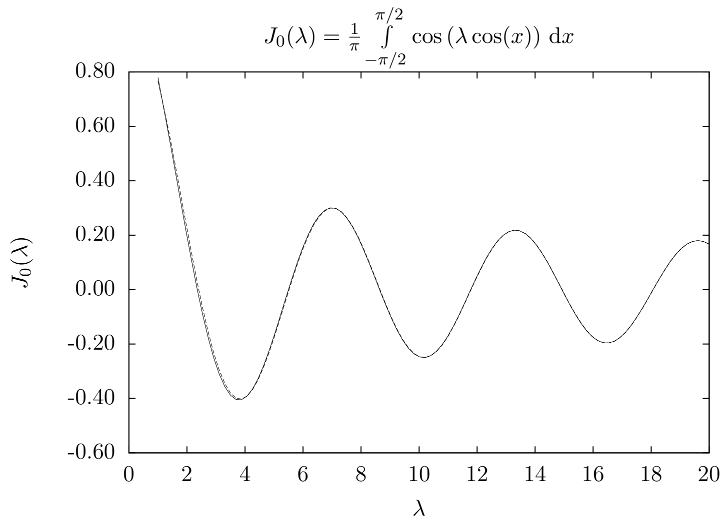

Figure 3: Asymptotics Eq. (**??**) (solid line) compared to numerically evaluated Eq. (10) (dashed line) for $1 \le x \le 20$.

The stationary point of the phase factor is at $\theta = 0$. Only small θ contribute to the integral. Therefore.

$$\cos\theta \approx 1 - \frac{\theta^2}{2}. \tag{21}$$

$$J_0(x) \sim \frac{1}{\pi}\,\mathrm{Re}\,e^{ix}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\frac{x\theta^2}{2}}\mathrm{d}\theta \sim \frac{1}{\pi}\sqrt{\frac{2}{x}}\,\mathrm{Re}\,e^{ix}\int_{-\infty}^{\infty} e^{-i\frac{x}{2}\theta^2}\mathrm{d}\left(\sqrt{\frac{x}{2}}\theta\right) \tag{22}$$

$$= \frac{1}{\pi}\sqrt{\frac{2}{x}}\,\mathrm{Re}\left(e^{ix}\sqrt{\pi}e^{-i\frac{\pi}{4}}\right) = \boxed{\sqrt{\frac{2\pi}{x}}\cos\left(x - \frac{\pi}{4}\right)} \tag{23}$$

References

[1] Lorella M. Jones. *An introduction to mathematical methods of physics*. Benjamin Cummings, 1979.

[2] Carl M. Bender and Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. Springer Verlag, 1999.