Summation of series: Sommerfeld-Watson transformation

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Contour integration can be used to sum infinite series. Suppose $f(z)$ is a function which is analytic at the integers $z = n$, $n = 0, \pm 1, \pm 2, \ldots$ and tends to zero at least as fast as

$$\frac{1}{|z|^2} \quad \text{as} \quad |z| \to \infty. \tag{1}$$

Otherwise $f(z)$ is an arbitrary function.

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Our goal is derive a usable expression for the following sum:

$$S = \sum_{n=-\infty}^{\infty} f(n) \tag{2}$$

Let's consider the function

$$F(z) = \pi f(z) \frac{\cos(\pi z)}{\sin(\pi z)}. \tag{3}$$

This function has simple poles at $z = n$, $n = 0, \pm 1, \pm 2, \ldots$ with residues

$$\mathrm{Res}\,(F(z), z = n) = \lim_{z \to n} \frac{\pi f(z) \cos(\pi z)}{\frac{\mathrm{d}}{\mathrm{d}z} \sin(\pi z)} = \lim_{z \to n} \frac{\pi f(z) \cos(\pi z)}{\pi \cos(\pi z)} = f(n).$$

$$\mathrm{Res}\,(F(z), z = n) = f(n). \tag{4}$$

The poles of $f(z)$, $z = z_i$, are also poles of $F(z)$ that are different from the poles at $z = n$.

The integral of $F(z)$ over the circle centered at the origin and of radius $R \to \infty$ is zero due to conditions Eq. (1).

$$\oint_C F(z)\mathrm{d}z = 0. \tag{5}$$

On the other hand

$$\oint_C F(z)\mathrm{d}z = 2\pi i \left[\sum_{n=-\infty}^{\infty} \mathrm{Res}\left(F(z), z=n\right) + \sum_i \mathrm{Res}\left(F(z), z=z_i\right) \right]. \tag{6}$$

Comparing Eq. (6) and Eq. (5), and using Eq. (4) we arrive at the following result:

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_i \mathrm{Res}\left(F(z), z=z_i\right), \tag{7}$$

where the summation on the right is over the poles of $f(z)$ and $F(z)$ is given by Eq. (3)

Example 1. Let's evaluate the following sum:

$$S(a) = \sum_{n=0}^{\infty} \frac{1}{n^2+a^2}, \tag{8}$$

where a is a real positive parameter.

1. Convert the summation over the integer n from $[0, \infty)$ to $(-\infty, \infty)$. Since the terms in the sum are even functions of n,

$$S(a) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} + \frac{1}{2a^2}, \tag{9}$$

where the last term on the right of Eq. (9) takes into account that the term with $n=0$ appears in the summation only once.

2. The sum in Eq. (9) has the form of Eq. (2). Therefore we can use the developed technique summarized in Eqs. (7), (3). Here

$$f(z) = \frac{1}{z^2+a^2}. \tag{10}$$

$f(z)$ has two simple poles at

$$z_{1,2} = \pm ia. \tag{11}$$

$$F(z) = \frac{\pi}{z^2+a^2}\frac{\cos(\pi z)}{\sin(\pi z)}. \tag{12}$$

Therefore,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = -\mathrm{Res}\left(\frac{\pi}{z^2+a^2}\frac{\cos(\pi z)}{\sin(\pi z)}, z=ia\right) - \mathrm{Res}\left(\frac{\pi}{z^2+a^2}\frac{\cos(\pi z)}{\sin(\pi z)}, z=-ia\right). \tag{13}$$

The first term on the right,

$$\mathrm{Res}\left(\frac{\pi}{z^2+a^2}\frac{\cos(\pi z)}{\sin(\pi z)}, z=ia\right) = \lim_{z\to ia}\left[\frac{\pi(z-ia)}{(z+ia)(z-ia)}\frac{\cos(\pi z)}{\sin(\pi z)}\right] = \frac{\pi}{2ia}\frac{\cos(i\pi a)}{\sin(i\pi a)}. \tag{14}$$

$$\cos(i\pi a) = \frac{1}{2}\left(e^{ii\pi a} + e^{-ii\pi a}\right) = \frac{1}{2}\left(e^{-\pi a} + e^{\pi a}\right) = \cosh(\pi a). \tag{15}$$

$$\sin(i\pi a) = \frac{1}{2i}\left(e^{ii\pi a} - e^{-ii\pi a}\right) = \frac{1}{2i}\left(e^{-\pi a} - e^{\pi a}\right) = i\sinh(\pi a). \tag{16}$$

Thus,

$$\text{Res}\left(\frac{\pi}{z^2+a^2}\frac{\cos(\pi z)}{\sin(\pi z)}, z = ia\right) = \frac{\pi}{2ia}\frac{\cosh(\pi a)}{i\sinh(\pi a)} = -\frac{\pi}{2a}\frac{\cosh(\pi a)}{\sinh(\pi a)}. \tag{17}$$

Similarly,

$$\text{Res}\left(\frac{\pi}{z^2+a^2}\frac{\cos(\pi z)}{\sin(\pi z)}, z = -ia\right) = \frac{\pi}{2ia}\frac{\cosh(\pi a)}{i\sinh(\pi a)} = -\frac{\pi}{2a}\frac{\cosh(\pi a)}{\sinh(\pi a)}. \tag{18}$$

Therefore, from Eqs. (13), (17), and (18)

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{a}\frac{\cosh(\pi a)}{\sinh(\pi a)}, \tag{19}$$

and, finally, from Eq. (9),

$$\sum_{n=0}^{\infty}\frac{1}{n^2+a^2} = \frac{1}{2a^2} + \frac{\pi}{2a}\frac{\cosh(\pi a)}{\sinh(\pi a)}. \tag{20}$$

Example 2. Let's evaluate the so called Basel sum:

$$B = \sum_{n=1}^{\infty}\frac{1}{n^2}. \tag{21}$$

1. Convert the summation over the integer n from $[1, \infty)$ to $(-\infty, \infty)$, with the exclusion of the term corresponding to $n = 0$. Since the therms in the sum are even functions of n,

$$B = \frac{1}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{1}{n^2}. \tag{22}$$

2. The sum in Eq. (22) almost has the form of Eq. (2). Therefore we can use the developed technique summarized in Eqs. (7), (3), with some adjustments. Here

$$f(z) = \frac{1}{z^2}. \tag{23}$$

$f(z)$ has second order pole at $z = 0$.

$$F(z) = \frac{\pi}{z^2}\frac{\cos(\pi z)}{\sin(\pi z)}. \tag{24}$$

$z = 0$ is the third order pole of $F(z)$. Therefore,

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2} = -\mathrm{Res}\left(\frac{\pi}{z^2}\frac{\cos(\pi z)}{\sin(\pi z)}, z = 0\right). \tag{25}$$

Using a computer algebra system, we find that the Laurent series for $F(z)$ is as following:

$$\frac{\pi}{z^2}\frac{\cos(\pi z)}{\sin(\pi z)} = \frac{1}{z^3} - \frac{\pi^2}{3z} - \frac{\pi^4 z}{45} - \frac{2\pi^6 z^3}{945} - \frac{\pi^8 z^5}{4725} + \ldots \tag{26}$$

Thus

$$\mathrm{Res}\left(\frac{\pi}{z^2}\frac{\cos(\pi z)}{\sin(\pi z)}, z = 0\right) = -\frac{\pi^2}{3}. \tag{27}$$

Finally,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2}\mathrm{Res}\left(\frac{\pi}{z^2}\frac{\cos(\pi z)}{\sin(\pi z)}, z = 0\right) = \frac{\pi^2}{6} \tag{28}$$

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Let's consider another class of sums:

$$S = \sum_{n=-\infty}^{\infty} (-1)^n f(n). \tag{29}$$

The following function,

$$F(z) = \frac{\pi f(z)}{\sin(\pi z)}. \tag{30}$$

This function has simple poles at $z = n$, $n = 0, \pm1, \pm2, \ldots$ with residues

$$\mathrm{Res}\,(F(z), z = n) = \lim_{z\to n} \frac{\pi f(z)}{\frac{\mathrm{d}}{\mathrm{d}z}\sin(\pi z)} = \lim_{z\to n} \frac{\pi f(z)}{\pi\cos(\pi z)} = \frac{f(n)}{\cos(\pi n)} = \frac{f(n)}{(-1)^n} = (-1)^n f(n).$$

$$\mathrm{Res}\,(F(z), z = n) = (-1)^n f(n). \tag{31}$$

The poles of $f(z)$, $z = z_j$, are also poles of $F(z)$ that are different from the poles at $z = n$.

Using the same reasoning that we used in Sec. 1, arrive at the expression:

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_j \mathrm{Res}\,(F(z), z = z_j), \tag{32}$$

where the summation on the right is over the poles of $f(z)$ and $F(z)$ is given by Eq. (30)

Example 3. Let's evaluate the following sum:

$$S(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + a^2}, \tag{33}$$

where a is a real positive parameter.

1. Convert the summation over the integer n from $[0, \infty)$ to $(-\infty, \infty)$. Since the terms in the sum are even functions of n,

$$S(a) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} + \frac{1}{2a^2}, \tag{34}$$

where the last term on the right of Eq. (34) takes into account that the term with $n = 0$ appears in the summation only once.

2. The sum in Eq. (34) has the form of Eq. (31). Therefore we can use the developed technique summarized in Eqs. (32), (30). Here

$$f(z) = \frac{1}{z^2 + a^2}. \tag{35}$$

$f(z)$ has two simple poles at

$$z_{1,2} = \pm ia. \tag{36}$$

$$F(z) = \frac{1}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}. \tag{37}$$

Therefore,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} = -\mathrm{Res}\left(\frac{1}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = ia\right) - \mathrm{Res}\left(\frac{1}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = -ia\right). \tag{38}$$

The first term on the right,

$$\mathrm{Res}\left(\frac{1}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = ia\right) = \lim_{z \to ia} \left[\frac{(z - ia)}{(z + ia)(z - ia)} \frac{\pi}{\sin(\pi z)}\right] = \frac{1}{2ia} \frac{\pi}{\sin(i\pi a)}. \tag{39}$$

Using Eq. (16)

$$\mathrm{Res}\left(\frac{1}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = ia\right) = \frac{1}{2ia} \frac{\pi}{i \sinh(\pi a)} = -\frac{1}{2a} \frac{\pi}{\sinh(\pi a)}. \tag{40}$$

Similarly,

$$\mathrm{Res}\left(\frac{1}{z^2 + a^2} \frac{\pi}{\sin(\pi z)}, z = -ia\right) = \frac{1}{2ia} \frac{\pi}{i \sinh(\pi a)} = -\frac{1}{2a} \frac{\pi}{\sinh(\pi a)}. \tag{41}$$

Therefore, from Eqs. (38), (40), and (41)

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} = \frac{\pi}{a} \frac{1}{\sinh(\pi a)}, \tag{42}$$

and, finally, from Eq. (34),

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{1}{2a^2} + \frac{\pi}{2a}\frac{1}{\sinh(\pi a)}. \tag{43}$$

Example 4.

$$S(x) = \frac{\sin x}{1+a^2} - \frac{2\sin 2x}{4+a^2} + \frac{3\sin 3x}{9+a^2} - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \sin nx}{n^2+a^2} \tag{44}$$

$$S(x) = -\frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} n \sin nx}{n^2+a^2} \tag{45}$$

$$f(z) = \frac{z\sin zx}{z^2+a^2} \tag{46}$$

$$F(z) = \frac{z\sin zx}{z^2+a^2}\frac{\pi}{\sin(\pi z)} \tag{47}$$

$$S(x) = \frac{1}{2}\mathrm{Res}\left(\frac{z\sin zx}{z^2+a^2}\frac{\pi}{\sin(\pi z)}, z=ia\right) + \frac{1}{2}\mathrm{Res}\left(\frac{z\sin zx}{z^2+a^2}\frac{\pi}{\sin(\pi z)}, z=-ia\right) \tag{48}$$

$$\mathrm{Res}\left(\frac{z\sin zx}{z^2+a^2}\frac{\pi}{\sin(\pi z)}, z=ia\right) = \frac{ia\sin iax}{2ia}\frac{\pi}{\sin(\pi ia)} = \frac{\pi}{2}\frac{\sinh ax}{\sinh a\pi} \tag{49}$$

$$\mathrm{Res}\left(\frac{z\sin(zx)}{z^2+a^2}\frac{\pi}{\sin(\pi z)}, z=-ia\right) = \frac{(-ia)\sin(-iax)}{(-2ia)}\frac{\pi}{\sin(-\pi ia)} = \frac{\pi}{2}\frac{\sinh(ax)}{\sinh(a\pi)} \tag{50}$$

$$S(x) = \frac{\pi}{2}\frac{\sinh(ax)}{\sinh(a\pi)} \tag{51}$$