Show all your work and indicate your reasoning in order to receive the most credit. Present your answers in *low-entropy* form. Write your name on the problems page and staple it together with your solutions.

Name: ______________________________

Date: ______________________________

Question:	1	2	Total
Points:	25	45	70
Score:			

Integral to stump a computer algebra system

1. (a) (15 points) Construct an integral that you can evaluate analytically but a computer algebra can not. Use the method described in the handout "The integral that stumped Feynman".

   (b) (10 points) To verify your result evaluate your integral and your answer numerically.

   Use a computer algebra system for auxiliary calculations (e.g. finding the real or imaginary parts of a complex expression) and numerics. Be elegant!

Polaron problem

2. An electron in a crystal polarizes the lattice in its neighborhood. The interaction changes the energy of the electron, and furthermore, when the electron moves the polarization state must move with it. An electron moving with its accompanying distortion of the lattice is called a *polaron*. It has an effective mass higher than that of the electron. We wish to compute the energy of such an electron.

   The kinetic energy of the "free" electron which moves with the linear momentum $\vec{\mathbf{p}}$ is

$$E_o(p) = \frac{p^2}{2}. \tag{1}$$

   Here we use the dimensionless units where mass of the electron, $m = 1$. The change of the energy of the electron due to interaction with the crystal is as following:

$$\Delta E(p) = -\frac{\alpha}{\sqrt{2}\pi^2} \iiint_{-\infty}^{\infty} \frac{\mathrm{d}^3\mathbf{k}}{(k^2 - 2\vec{\mathbf{p}}\cdot\vec{\mathbf{k}} + 2)\, k^2}, \tag{2}$$

   where α is a constant describing the electron-crystal interaction, $k = |\vec{\mathbf{k}}|$, $p = |\vec{\mathbf{p}}|$, and the integration is in three dimensions:

$$\mathrm{d}^3\mathbf{k} = \mathrm{d}k_x\, \mathrm{d}k_y\, \mathrm{d}k_z \tag{3}$$

in cartesian coordinates;

$$d^3\mathbf{k} = k^2 dk\ \sin\theta\, d\theta\, d\varphi \tag{4}$$

in spherical coordinates.

Your task is to evaluate the integral in Eq. (2).

(a) (10 points) To evaluate the integral you are going to use the identity known as *Feynman parametrization* [**?**].

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2}. \tag{5}$$

Show that Eq. (5) is indeed correct by evaluating the integral on the right (it is an elementary integral).

(b) (10 points) Convert the triple integral Eq. (2) into a quadruple integral using identity Eq. (5). Simplify the integral by shifting the integration variable

$$\vec{\mathbf{k}} - x\vec{\mathbf{p}} \quad \longrightarrow \quad \vec{\mathbf{k}} \tag{6}$$

(c) (15 points) For the next step you are going to need the following integral.

$$I = \iiint\limits_{-\infty}^{\infty} \frac{d^3\mathbf{k}}{(k^2+\gamma)^2}, \tag{7}$$

where γ is independent of k, $\gamma = \gamma(x)$. Evaluate the integral in polar coordinates. Use the method of residues. Sketch your integration contour in indicate the order of the pole(s) inside.

Selfcheck: $I = \frac{\pi^2}{\sqrt{\gamma}}$.

(d) (5 points) Finally, you are going to need the integral

$$\int_0^1 \frac{dx}{\sqrt{\gamma}} = \int_0^1 \frac{dx}{\sqrt{ax - bx^2}}, \tag{8}$$

where a and b are real positive parameters. This is an elementary integral. To evaluate the integral into the most elegant form, start with the introduction of the new integration variable, u:

$$u^2 = x: \quad 0 \le u \le 1, \quad dx = 2u\,du, \quad u = \sqrt{x}. \tag{9}$$

(e) (5 points) Collect all factors and write the final expression for the total energy of the polaron, $E(p)$.