Show all your work and indicate your reasoning in order to receive the most credit. Present your answers in *low-entropy* form. Write your name on the problems page and staple it together with your solutions.

Name: ______________________________

Date: ______________________________

Question:	1	2	3	4	5	6	7	Total
Points:	10	5	5	5	10	5	5	45
Score:								

Incoherent self-induced transparency

Consider photosensitive molecules distributed uniformly in a transparent liquid irradiated by a light beam. As the beam penetrates the liquid, it is absorbed by the molecules, which decompose and become transparent to light. Let the liquid occupy the region $X > 0$ and the light beam propagate in the positive X direction. Let $I(X,T)$ and $N(X,T)$ denote the light intensity and the concentration of molecules, respectively. The problem is to find $I(X,T)$ and $N(X,T)$ as functions of coordinate, X, and time, T.

The amount of light absorbed is proportional to the product of the light intensity and the concentration of the photosensitive molecules:

$$\frac{\partial I}{\partial X} = -\alpha N I, \tag{1}$$

where α is the absorption coefficient. On the other hand, the rate of decomposition of the molecules is proportional to the rate of light absorption:

$$\frac{\partial N}{\partial T} = \beta \frac{\partial I}{\partial X}, \tag{2}$$

where β is the decay constant.

Let the initial concentration of molecules $N(X,0) = N_0$ and the intensity of the light beam as it enters the fluid $I(0,T) = I_0$. Let's assume that the beam is turned on at $T = 0$. The solution of Eq. (1) at $T = 0$ gives the initial distribution of light beam intensity:

$$I(X,0) = I_0 e^{-\alpha N_0 X}. \tag{3}$$

The suggested steps to solve the problem are as following:

1. (10 points) Rewrite Eqs. (1)- (2) in dimensionless units: introduce dimensionless intensity of the light beam,

$$i = \frac{I}{I_0}, \quad 0 < i < 1,$$

and dimensionless concentration,

$$n = \frac{N}{N_0}, \quad 0 < n < 1.$$

Notice that the combination

$$\lambda = \alpha N_0$$

has the dimension of (length)$^{-1}$. Introduce the dimensionless coordinate

$$x = X\lambda.$$

Notice that the combination

$$\tau = \frac{1}{\alpha\beta I_0}$$

has the dimension of time. Introduce dimensionless time-like variable

$$t = \frac{T}{\tau}.$$

What are the dimensionless versions of Eqs. (1) (2)? What is the initial dimensionless concentration, $n(x, 0)$? What is the dimensionless intensity of the beam as it enters the fluid, $i(0, t)$?

2. (5 points) Rewrite the dimensionless version of Eq. (1) into the equation for $\log(i)$
3. (5 points) Differentiate the new equation with respect to dimensionless time and use the dimensionless version of Eq. (2) to exclude concentration of molecules n. You obtained a PDE for $i(x, t)$.
4. (5 points) Integrate both sides of the PDE with respect to x. Chose the integration constant such that the intensity of the beam at $x = 0$ is always the constant. You obtained an ODE for $i(x, t)$.
5. (10 points) Solve the first order ODE for $i(x, t)$. Find the integration constant (which is actually a function of x) comparing your solution with Eq. (3), which in the dimensionless units is $i(x, 0) = e^{-x}$.
6. (5 points) Find $n(x, t)$ using the dimensionless version of Eq. (1).
7. (5 points) Plot $i(x, t)$ and $n(x, t)$ for $0 < x < 10$ and $t = 0, 2, 4, \ldots, 12$.