Weakly nonlinear oscillators

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1 Oscillator with nonlinear friction

Let's consider the following second order non-linear differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \mu\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3 + x = 0, \quad \mu > 0. \tag{1}$$

The equation models a non-conservative system in which energy is dissipated. The parameter μ is a positive scalar indicating the nonlinearity and the rate of the energy losses.

1.1 Numerical integration

To solve Eq. (1) numerically, let's write Eq. (1) as a system of first order differential equations,

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -\mu y^3 - x, \end{cases} \tag{2}$$

The results of numerical integration of Eqs. (2) are presented in Fig. 1.

1.2 Small nonlinearity – the method of averaging

To obtain an approximate analytic solution of Eq. (1), we use a powerful method called the *method of averaging*. It is applicable to equations of the following general form:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x = \mu F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right), \tag{3}$$

where in our case

$$F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = -\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3. \tag{4}$$

We seek a solution to Eq. (3) in the form:

$$x = a(t)\cos\left(t + \psi(t)\right), \tag{5}$$

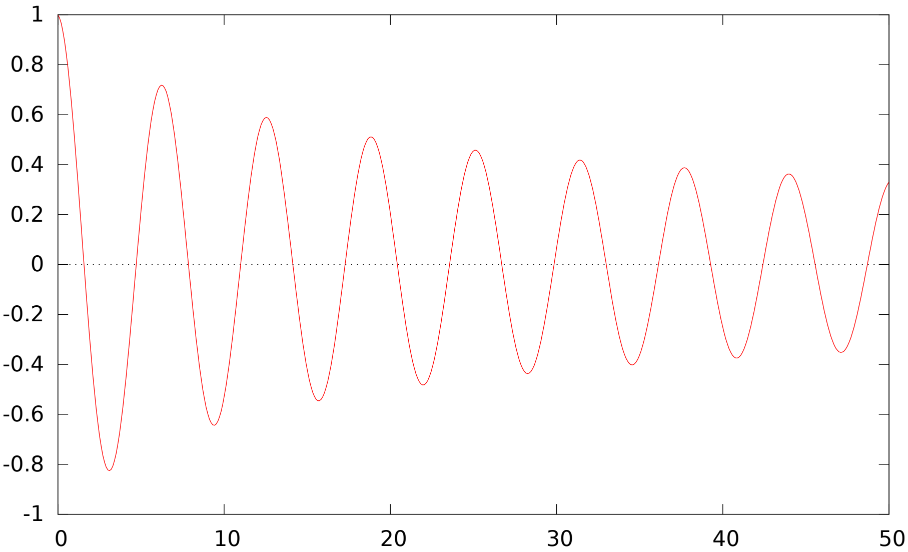


Figure 1: Typical solution of Eq. (1) for small values of μ; $\mu = 0.2$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -a(t)\sin\left(t + \psi(t)\right). \tag{6}$$

The motivation for this ansatz is that when μ is zero, Eq. (3) has its solution of the form Eq. (5) with a and ψ constants. For small values of μ we expect the same form of the solution to be approximately valid, but now a and ψ are expected to be slowly varying functions of t.

Differentiating Eq. (5) and requiring Eq. (6) to hold, we obtain:

$$\dot{a}\cos\left(t + \psi(t)\right) - a\dot{\psi}\sin\left(t + \psi(t)\right) = 0. \tag{7}$$

where

$$\dot{a} \equiv \frac{\mathrm{d}a}{\mathrm{d}t}, \quad \dot{\psi} \equiv \frac{\mathrm{d}\psi}{\mathrm{d}t}. \tag{8}$$

Differentiating Eq. (6) and substituting the result into Eq. (3) gives

$$-\dot{a}\sin\left(t + \psi\right) - a\dot{\psi}\cos\left(t + \psi\right) = \mu a^3 \sin^3\left(t + \psi\right). \tag{9}$$

Solving Eqs. (7) and (9) for $\dot{a}$ and $\dot{\psi}$, we obtain:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\mu a^3 \sin^4\left(t + \psi\right) \tag{10}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\mu a^2 \sin^3\left(t + \psi\right)\cos(t + \psi). \tag{11}$$

So far our treatment has been exact and is essentially the procedure of variation of parameters which is used to obtain particular solutions to nonhomogenous linear differential equations.

Now we introduce the following approximation: since μ is small, $\frac{\mathrm{d}a}{\mathrm{d}t}$ and $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ are also small. Hence $a(t)$ and $\psi(t)$ are slowly varying functions of t. Thus over one cycle of oscillations the quantities $a(t)$

and $\psi(t)$ on the right hand sides of Eqs. (10) and (11) can be treated as nearly constant, and thus these right hand sides may be replaced by their averages:

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \ldots \tag{12}$$

Eqs. (10) and (11) become

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\mu\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi a^3 \sin^4(\phi) \tag{13}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\mu\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi a^2 \sin^3(\phi)\cos(\phi) \tag{14}$$

The right hand side of Eq. (14) is zero. The averaging in Eq. (13) can be done using the following trigonometric identities:

$$\sin^2(\phi) = \frac{1}{2}\left(1-\cos(2\phi)\right),$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \cos^2(n\phi) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \sin^2(n\phi) = \frac{1}{2}, \quad n = 1, 2, \ldots$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \cos(n\phi) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \sin(n\phi) = 0, \quad n = 1, 2, \ldots$$

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \sin^4(\phi) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \left(\frac{1}{2}\left(1-\cos(2\phi)\right)\right)^2 = \\ &= \frac{1}{4}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \left(1 - 2\cos(2\phi) + \cos^2(2\phi)\right) = \\ &= \frac{1}{4}\left(1+\frac{1}{2}\right) = \frac{3}{8}\end{aligned} \tag{15}$$

The averaged equations of motion are as following:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\mu\frac{3}{8}a^3 \tag{16}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = 0 \tag{17}$$

The solution of Eq. (17) is

$$\psi = 0. \tag{18}$$

Eq. (16) can be solved by separating the variables:

$$\frac{\mathrm{d}a}{a^3} = -\frac{3}{8}\mu \mathrm{d}t \quad \rightarrow \quad \frac{1}{a^2(t)} = \frac{3}{4}\mu t + \frac{1}{a^2(0)} \quad \rightarrow \quad a(t) = \frac{1}{\sqrt{\frac{3}{4}\mu t + \frac{1}{a^2(0)}}}, \tag{19}$$

where $a(0)$ is the amplitude of oscillations at $t = 0$. Finally,

$$x(t) = \frac{\cos(t)}{\sqrt{\frac{3}{4}\mu t + \frac{1}{a^2(0)}}} \tag{20}$$

2 Van der Pol oscillator

The second order non-linear autonomous differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \mu\left(x^2 - 1\right)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0, \quad \mu > 0 \tag{21}$$

is called the van der Pol equation. It describes many physical systems collectively called *van der Pol oscillators*. The equation models a non-conservative system in which energy is added to and subtracted from the system, resulting in a periodic motion called a *limit cycle*. The parameter mu is a positive scalar indicating the nonlinearity and the strength of the damping. The sign of the damping term in Eq. (21), $\left(x^2 - 1\right)\frac{\mathrm{d}x}{\mathrm{d}t}$ changes, depending upon whether $|x|$ is larger or smaller than one.

2.1 Numerical integration

Let's write Eq. (21) as a first order system of differential equations,

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -\mu\left(x^2 - 1\right)\dfrac{\mathrm{d}x}{\mathrm{d}t} - x, \end{cases} \tag{22}$$

The results of numerical integration of Eqs. (22) are presented in Figs. 2–3.

Numerical integration of Eq. (22) shows that every initial condition (except $x = 0$, $\dot{x} = 0$) approaches a unique periodic motion. The nature of this limit cycle is dependent on the value of μ. For small values of μ the motion is nearly sinusoidal.

Numerical integration shows that the limit cycle is a closed curve enclosing the origin in the x-y phase plane. From the fact that Eqs. (22) are invariant under the transformation $x \to -x$, $y \to -y$, we may conclude that the curve representing the limit cycle is point symmetric about the origin.

2.2 Small nonlinearity – the method of averaging

In order to obtain information regarding the approach to the limit cycle, we use a powerful method called the *method of averaging*. It is applicable to equations of the following general form:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x = \mu F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right), \tag{23}$$

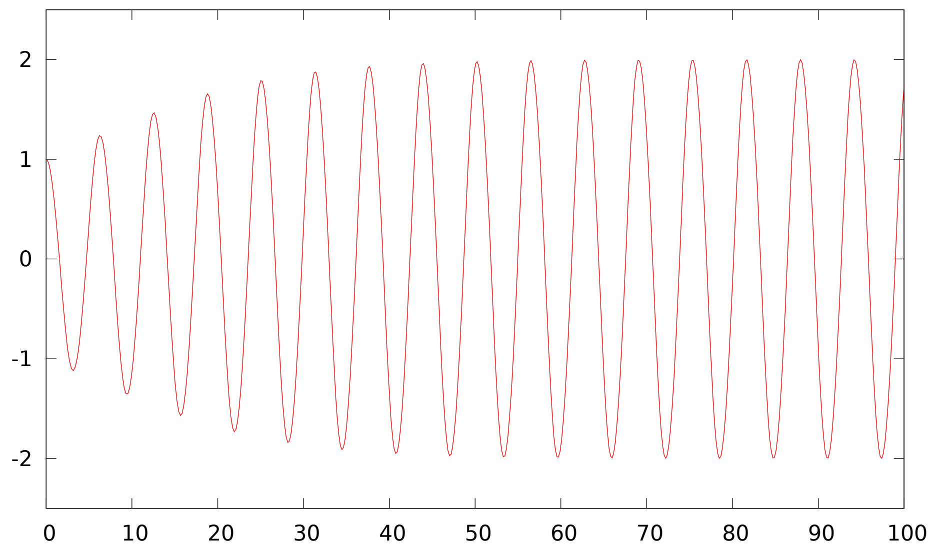
where in our case

$$F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = -\left(x^2 - 1\right)\frac{\mathrm{d}x}{\mathrm{d}t}. \tag{24}$$

We seek a solution to Eq. (23) in the form:

$$x = a(t)\cos\left(t + \psi(t)\right), \tag{25}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -a(t)\sin\left(t + \psi(t)\right). \tag{26}$$



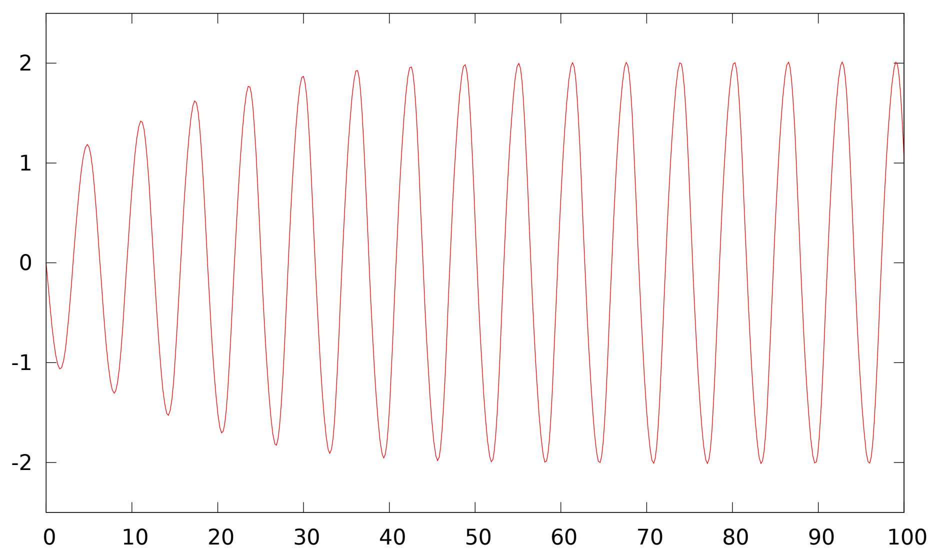


Figure 2: Typical solution of van der Pol equation for small values of μ; top graph – $x(t)$, bottom graph – $\dot{x}(t)$; $\mu = 0.1$

Our motivation for this ansatz is that when μ is zero, Eq. (23) has its solution of the form Eq. (25) with a and ψ constants. For small values of μ we expect the same form of the solution to be approximately valid, but now a and ψ are expected to be slowly varying functions of t. Differentiating Eq. (25) and requiring Eq. (26) to hold, we obtain:

$$\frac{\mathrm{d}a}{\mathrm{d}t}\cos\left(t+\psi(t)\right) - a\frac{\mathrm{d}\psi}{\mathrm{d}t}\sin\left(t+\psi(t)\right) = 0. \tag{27}$$

Differentiating Eq. (26) and substituting the result into Eq. (23) gives

$$-\frac{\mathrm{d}a}{\mathrm{d}t}\sin\left(t+\psi\right) - a\frac{\mathrm{d}\psi}{\mathrm{d}t}\cos\left(t+\psi\right) = \mu F\left(a(t)\cos\left(t+\psi\right), -a(t)\sin\left(t+\psi\right)\right). \tag{28}$$

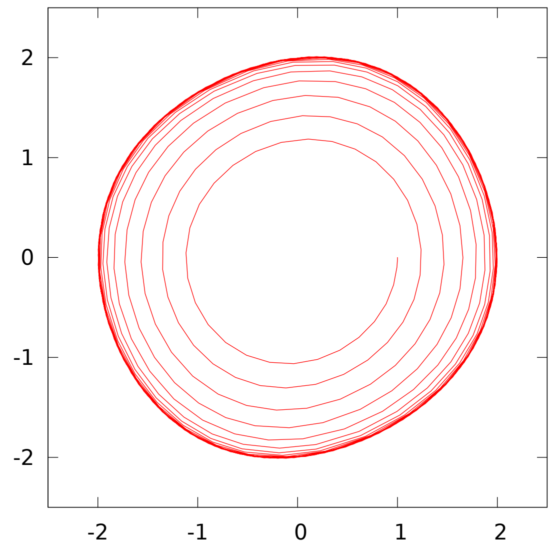


Figure 3: Solution of van der Pol equation for small values of μ.

Solving Eqs. (27) and (28) for $\frac{\mathrm{d}a}{\mathrm{d}t}$ and $\frac{\mathrm{d}\psi}{\mathrm{d}t}$, we obtain:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\mu F\left(a(t)\cos\left(t+\psi\right), -a(t)\sin\left(t+\psi\right)\right)\sin(t+\psi) \tag{29}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\frac{\mu}{a} F\left(a\cos\left(t+\psi\right), -a\sin\left(t+\psi\right)\cos(t+\psi\right), \tag{30}$$

where

$$F\left(\ldots\right) = a\left(a^2\cos^2\left(t+\psi\right)-1\right)\sin\left(t+\psi\right). \tag{31}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\mu a\left(a^2\cos^2\left(t+\psi\right)-1\right)\sin^2\left(t+\psi\right) \tag{32}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\mu\left(a^2\cos^2\left(t+\psi\right)-1\right)\sin\left(t+\psi\right)\cos\left(t+\psi\right) \tag{33}$$

So far our treatment has been exact and is essentially the procedure of variation of parameters which is used to obtain particular solutions to nonhomogenous linear differential equations.

Now we introduce the following approximation: since μ is small, $\frac{\mathrm{d}a}{\mathrm{d}t}$ and $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ are also small. Hence $a(t)$ and $\psi(t)$ are slowly varying functions of t. Thus over one cycle of oscillations the quantities $a(t)$ and $\psi(t)$ on the right hand sides of Eqs. (32) and (33) can be treated as nearly constant, and thus these right hand sides may be replaced by their averages:

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\ldots \tag{34}$$

Eqs. (32) and (33) become

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\mu\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi a\left(a^2\cos^2\left(\phi\right)-1\right)\sin^2\left(\phi\right) \tag{35}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\mu\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\left(a^2\cos^2\left(\phi\right)-1\right)\sin\left(\phi\right)\cos\left(\phi\right) \tag{36}$$

The right hand side of Eq. (36) is zero. The averaging in Eq. (35) can be done using the following trigonometric identities:

$$\cos^2(\phi) = \frac{1}{2}\left(1+\cos(2\phi)\right), \quad \sin^2(\phi) = \frac{1}{2}\left(1-\cos(2\phi)\right),$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \cos^2(n\phi) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \sin^2(n\phi) = \frac{1}{2}, \quad n = 1,2,\ldots$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \cos(n\phi) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \sin(n\phi) = 0, \quad n = 1,2,\ldots$$

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \cos^4(\phi) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \left(\frac{1}{2}\left(1+\cos(2\phi)\right)\right)^2 = \\ &= \frac{1}{4}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \left(1+2\cos(2\phi)+\cos^2(2\phi)\right) = \\ &= \frac{1}{4}\left(1+\frac{1}{2}\right) = \frac{3}{8} \end{aligned} \tag{37}$$

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi a\left(a^2\cos^2(\phi)-1\right)\sin^2(\phi) &= \frac{a^3}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \cos^2(\phi)\left(1-\cos^2(\phi)\right) \\ &\quad -\frac{a}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \sin^2(\phi) \\ &= \frac{a^3}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \left(\cos^2(\phi)-\cos^4(\phi)\right) - \frac{a}{2} \\ &= a^3\left(\frac{1}{2}-\frac{3}{8}\right) - \frac{a}{2} = \frac{1}{8}a(a^2-4) \end{aligned} \tag{38}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\mu}{8}a(4-a^2) \tag{39}$$

Eq. (39) can be solved separating variables:

$$\frac{\mathrm{d}a}{a(2-a)(2+a)} = \frac{1}{8}\mu \mathrm{d}t \tag{40}$$

$$\frac{1}{a(2-a)(2+a)} = -\frac{1}{4}\frac{1}{a} + \frac{1}{8}\frac{1}{2-a} + \frac{1}{8}\frac{1}{2+a} \tag{41}$$

$$-2\frac{\mathrm{d}a}{a} + \frac{\mathrm{d}a}{2-a} + \frac{\mathrm{d}a}{2+a} = \mu \mathrm{d}t \tag{42}$$

$$-2\frac{\mathrm{d}a}{a} - \frac{\mathrm{d}(2-a)}{2-a} + \frac{\mathrm{d}(2+a)}{2+a} = \mu \mathrm{d}t \tag{43}$$

$$-\mathrm{d}\log\left(a^2\right) - \mathrm{d}\log(2-a) + \mathrm{d}\log(2+a) = \mu \mathrm{d}t \tag{44}$$

$$\log\left(\frac{a+2}{a^2(2-a)}\right) = \mu(t-t_0), \tag{45}$$

$$\frac{a+2}{a^2(2-a)} = e^{\mu(t-t_0)}. \tag{46}$$

When $a \approx 2$,

$$a \approx 2 - e^{-\mu(t-t_0)} \tag{47}$$