Lotka-Volterra predator-prey model

Course material for PHYS2200 class

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A classical model in mathematical ecology is the Lotka-Volterra predator-prey model. Consider a simple ecosystem consisting of rabbits that have an infinite supply of food and foxes that prey on the rabbits for their food. This is modeled by a pair of nonlinear, 1st-order differential equations:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 2r - \alpha r f, \tag{1}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -f + \alpha r f, \tag{2}$$

where t is time, $r(t)$ is the number of rabbits, $f(t)$ is the number of foxes, and α is a positive constant. If $\alpha = 0$, the two populations do not interact, the rabbits multiply, and the foxes die off from starvation. If $\alpha > 0$, the foxes encounter the rabbits with a probability that is proportional to the product of their numbers. Such an encounter results in a decrease in the number of rabbits and (for less obvious reasons) an increase in the number of foxes.

The solutions to this nonlinear system cannot be expressed in terms of other known functions; the equations must be solved numerically. It turns out that the solutions are always periodic, with a period that depends on the initial conditions. In other words, for any $r(0)$ and $f(0)$, there is a value $t = t_p$ when both populations return to their original values. Consequently, for all t, $r(t+t_p) = r(t)$, $f(t+t_p) = f(t)$.

A. Compute the solution with $r(0) = 300$, $f(0) = 150$, and $\alpha = 0.01$. You should find that t_p is close to 5. Make two plots, one of r and f as functions of t and one a phase plane plot with r as one axis and f as the other.

B. If we divide Eq. (1) by Eq. (2) we obtain a single equation for $r = r(f)$ (or $f = f(r)$). Solve the equation using separation of variables and obtain an invariant – the combination of r and f that is constant in time. Modifying the program you wrote in part (A) and check the value of this combination.

C. The system of Eqs. (1)–(2) has a stable equilibrium point when the numbers of rabbits, r_0, and foxes, f_0, do not change, i.e.

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}f}{\mathrm{d}t} = 0. \tag{3}$$

Find $r_0(\alpha)$ and $f_0(\alpha)$. Modify the program you wrote in part (A) and numerically check for $\alpha = \frac{1}{100}$ that if the population numbers have these initial values, they do not change.