Name: ______________________________

Date: ______________________________

Question:	1	2	3	Total
Points:	40	40	20	100
Score:				

1. (40 points) **Volume of 5-dimensional superellipsoids**

   (a) Use Monte Carlo integration to calculate the volumes, $V_5(n)$, of the following geometrical objects (called superellipsoids) in 5-dimensional space:

   $$|x_1|^n + |x_2|^n + |x_3|^n + |x_4|^n + |x_5|^n = 1, \quad \text{for} \quad n = 1, 2, \ldots, 9. \tag{1}$$

   Here x_i are the cartesian coordinates.

   (b) Estimate statistical errors of your calculations. Adjust the number of randomly generated 5-dimensional points such that the error estimations are less than 0.03.

   (c) Plot the graph $V_5(n)$, volume of the superellipsoid vs. superellipsoid parameter, n. Prepare gnuplot scripts to generate a png graph from your data file.

   (d) Write README.md in markdown that includes all graphs that you generated with an explanation of your work. You may edit README.md directly on github (to have an immediate preview).

   Your code must be elegant, well formatted, and reasonably commented. Make sure you use valgrind to check your code for memory leaks. In the spirit of reproducible research approach, compilation of your code, actual calculations, and graphs generation should be controlled by your Makefile.

   Upload your code, Makefile, .gitignore, README.md, your graphics files to GitHub and provide the link to the project:

   https/github.com/ ______________________________

   Hint: to check whether your calculations make sense, note that a regular 5-dimensional octahedron is a special case of superellipsoid Eq. (1) when $n = 1$; thus $V_5(1) = \frac{2^5}{5!}$. Furthermore, when $n = 2$ Eq. (1) is the equation for a 5-dimensional unit sphere; thus $V_5(2) = \frac{\pi^{\frac{5}{2}}}{\Gamma(\frac{5}{2}+1)}$, where $\Gamma(x)$ is gamma function; $\Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi}$. Finally, for large n the superellipsoid Eq. (1) is approaching the cube with edge length equal to 2 (why?); thus $V(n) \to 2^5$ for large $n \gg 1$.

2. (40 points) A mechanical system consists of a rigid pendulum (a massless rigid rod of length l and a point mass at its end) whose suspension point is harmonically vibrating along the vertical direction with the frequency ω and the amplitude a. The equation of motion describing such a system is as following.

$$\ddot{\phi} + \omega_0^2 \left(1 - \frac{a}{l}\frac{\omega^2}{\omega_0^2}\sin(\omega t)\right)\sin(\phi) = 0. \tag{2}$$

Here ϕ is the angle between the pendulum's rod and the vertical "down" direction and $\omega_0 = \sqrt{\frac{l}{g}}$ is the frequency of pendulum's natural oscillations.

An analysis of Eq. (2) predicts that for sufficient large value of ω and a the vertical "up" position ($\phi = \pi$) of the pendulum is stable. Your task is to check this result numerically.

(a) Rewrite Eq. (2) in the dimensionless form.

$$\ddot{\phi} + \left(1 - \frac{a}{l}\Omega^2 \sin(\Omega\tau)\right)\sin(\phi) = 0, \tag{3}$$

where

$$\Omega = \frac{\omega}{\omega_0}.$$

Attach your derivation. Hint: introduce dimensionless time-like variable $\tau = \omega_0 t$.

(b) Write a program that solves Eq. (3) numerically for the following values of parameters,

$$\frac{a}{l} = 0.01$$

and the following initial conditions.

$$\phi(0) = 0.99\,\pi, \quad \dot{\phi}(0) = 0.$$

(c) To see the stable and the unstable oscillations around the vertical "up" direction, conduct your calculations for several different values of Ω between 100 and 200. Plot two graphs, $\phi(\tau)$, of "typical" stable and unstable oscillations.

(d) Determine by trial and error (with the relative error $\sim 5\%$) the critical value of Ω when the vertical up pendulum position looses its stability.

Record your result here: ______________________________

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Write README.md in markdown that includes all graphs that you generated that explains your work. You may edit README.md directly on github (to have an immediate preview).

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3. (20 points) Write a program that calculates the sum of the following series

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} \tag{4}$$

with the relative error less than $3 \times \varepsilon_{\mathrm{mach}}$. Here $\varepsilon_{\mathrm{mach}}$ is machine epsilon.

(a) Use `float` data type for **all** floating point variables and functions.

(b) Use the following code in your program to estimate $\varepsilon_{\mathrm{mach}}$:

```
float macheps = 1.;
while ((float) (1. + macheps) != 1.)
{
  macheps /= 2;
}
printf("macheps = %g\n", macheps);
```

(c) Start with $N = 10$ terms in the sum in Eq. (4) and double N sufficient number of times.

Your code must be elegant, well formatted, and reasonably commented. In the spirit of reproducible research approach, compilation of your code and actual calculations should be controlled by your Makefile.

Write README.md in markdown that explains of your work. You may edit README.md directly on github (to have an immediate preview).

Upload your code, Makefile, .gitignore, README.md to GitHub and provide the link to the project:

https/github.com/ ____