Glider flight

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1 Experimental facts

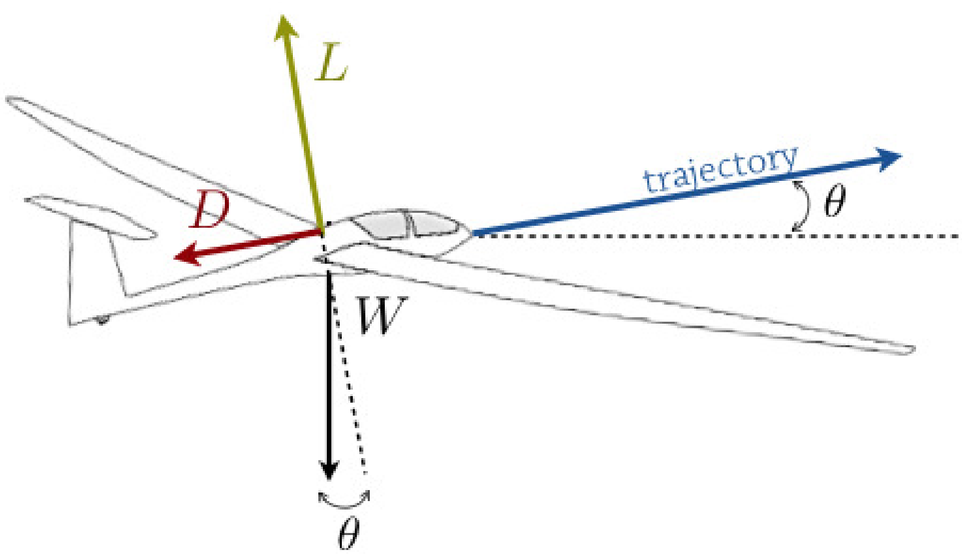

Figure 1: Forces acting on the glider: L, the lift, W, the force of gravity, $W = mg$, and D, the drag. θ is the angle between the instantaneous direction of the velocity (tangential direction to the trajectory) and the horizontal line.

The force of lift, L, created by the airflow around the wings is perpendicular to the trajectory, and the force of drag, D, is parallel to the trajectory. Both forces are expressed in terms of coefficients of lift and drag, C_L and C_D, respectively, that depend on the wing design and angle of attack – the angle between the wing chord and the flight path.

Lift and drag are proportional to a surface area of the wings, S, and the dynamic pressure, $1/2\rho V^2$, where ρ is the density of air, and V the forward velocity of the aircraft. The equations for lift and drag are:

$$L = \frac{1}{2}\rho V^2 C_L S \tag{1}$$
$$D = \frac{1}{2}\rho V^2 C_D S \tag{2}$$

The ration of lift to drag, L/D is called the aerodynamic efficiency of the aircraft.

$$R \equiv \frac{L}{D} = \frac{C_L}{C_D}. \tag{3}$$

2 Equations of motion

Newton's second law of motion applied to the motion tangential to the trajectory

$$m\,\frac{\mathrm{d}V}{\mathrm{d}t} = -mg\sin\theta - D, \tag{4}$$

where V is the speed of the glider, m is its mass, g is acceleration of gravity, D is the drag force given by Eq. (2).

The minus sign in front of the first term in the right hand side of Eq. (4) matches with our intuition: when θ is negative, the nose is pointing down and the plane accelerates due to gravity. When $\theta > 0$, the plane must fight against gravity.

In the normal direction, we have centripetal force, $m\frac{V^2}{r}$, where r is the instantaneous radius of curvature. After noticing that that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = V/r,$$

this can be expressed as $V\frac{d\theta}{dt}$, giving

$$m\,V\,\frac{\mathrm{d}\theta}{\mathrm{d}t} = -mg\cos\theta + L, \tag{5}$$

where L is the lift force given by Eq. (1).

$$m\frac{\mathrm{d}V}{\mathrm{d}t} = -mg\sin\theta - \frac{1}{2}\rho V^2 C_D S \tag{6}$$
$$mV\,\frac{d\theta}{dt} = -mg\cos\theta + \frac{1}{2}\rho V^2 C_L S \tag{7}$$

3 Glider trajectory

If we want to visualize the flight trajectories predicted by this model, we are going to need to integrate the spatial coordinates, which depend on both the forward velocity and the trajectory angle. The position of the glider on a vertical plane will be designated by coordinates (X, Y) with respect to an inertial frame of reference, and are obtained from:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = V\cos(\theta) \tag{8}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = V\sin(\theta). \tag{9}$$

Augmenting our original two differential equations by the two equations above, we have a system of four first-order differential equations to solve. We will use a time-stepping approach, like in the previous lesson. To do so, we do need initial values for every unknown:

$$V(0) = V_0 \quad \text{and} \quad \theta(0) = \theta_0 \tag{10}$$

$$X(0) = X_0 \quad \text{and} \quad Y(0) = Y_0 \tag{11}$$

4 Scaling the equations

It is often useful to reduce equations describing a physical system to dimensionless form, both for physical insight and for numerical convenience (i.e., to avoid dealing with very large or very small numbers in the computer). To do this for the equations of glider motion, we introduce dimensionless speed, time, and length variables.

To introduce the characteristic velocity, let's consider horizontal motion of the glider with a constant velocity:

$$\theta = 0, \quad V = v_t = const. \tag{12}$$

From Eqs. (6)-(7) we conclude that such motion is possible only when (a) the drag force acting on the glider is zero, i.e. $C_D = 0$, and (b) when the force of gravity is balanced by the lift force:

$$mg = \frac{1}{2}\rho v_t^2 C_L S. \tag{13}$$

The relation Eq. (13) introduces the characteristic velocity, so called trim velocity, v_t.

$$v_t = \sqrt{\frac{mg}{\frac{1}{2}\rho C_L S}}. \tag{14}$$

For the reference,

$$m = \frac{1}{2g}\rho v_t^2 C_L S. \tag{15}$$

From now on we are going to measure the speed of the glider in units of v_t, i.e. we introduce new dimensionless speed variable

$$v \equiv \frac{V}{v_t}. \tag{16}$$

The characteristic acceleration in the glider problem is acceleration of gravity, g. We can combine characteristic acceleration with characteristic velocity, v_t, to get a characteristic variable, t_0 with the dimension of time:

$$t_c = \frac{v_t}{g}. \tag{17}$$

t_c is the time for a free falling body starting from rest to gain the speed v_t.

From now on we are going to measure the time in units of t_c, i.e. we introduce new dimensionless time variable

$$\tau \equiv \frac{t}{t_c}, \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{t_c}\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{g}{v_t}\frac{\mathrm{d}}{\mathrm{d}\tau}. \tag{19}$$

We introduce the characteristic length as the product of v_t and t_c.

$$l_c = t_c v_t = \frac{v_t^2}{g}. \tag{20}$$

l_c is equal double the height for a free falling body starting from rest to gain the speed v_t.

We'll measure the X and Y coordinates of the glider in units of l_c:

$$x = \frac{X}{l_c}, \quad y = \frac{Y}{l_c}. \tag{21}$$

Substituting Eqs. (13), (15)–(17), and (19) into Eqs. (6)–(7), we obtain:

$$\frac{1}{2g}\rho v_t^2 C_L S v_t \frac{g}{v_t}\frac{\mathrm{d}v}{\mathrm{d}\tau} = -\frac{1}{2}\rho v_t^2 C_L S \sin\theta - \frac{1}{2}\rho v_t^2 v^2 C_D S \tag{22}$$

$$\frac{1}{2g}\rho v_t^2 C_L S v v_t \frac{g}{v_t}\frac{d\theta}{d\tau} = -\frac{1}{2}\rho v_t^2 C_L S \cos\theta + \frac{1}{2}\rho v_t^2 v^2 C_L S \tag{23}$$

Canceling common factors, arrive to the following system of ODEs:

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = -\sin\theta - \frac{v^2}{R}, \tag{24}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{\cos\theta}{v} + v. \tag{25}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = v\cos\theta \tag{26}$$
$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = v\sin\theta \tag{27}$$

5 Steady state flight

One way we can get a better handle on exactly how the behavior of the solutions depends on R is to examine what happens to solutions near the constant solution. If we look in the $\theta - v$ plane, this solution corresponds to a single point, which is often referred to as a "fixed point". We explicitly find the fixed point by noticing that whenever $\dot{\theta} = 0$ and $\dot{v} = 0$, we must have a constant solution:

$$-\sin\theta - \frac{v^2}{R} = 0 \tag{28}$$
$$-\cos\theta + v^2 = 0 \tag{29}$$

$$\sin^2\theta = \frac{v^4}{R^2} \tag{30}$$
$$\cos^2\theta = v^4 \tag{31}$$

$$v = \sqrt[4]{\frac{1}{1+\frac{1}{R^2}}} \tag{32}$$
$$\theta = -\sin^{-1}\sqrt{\frac{1}{1+R^2}} \tag{33}$$

So, we see that there is a fixed point for all values of R (since 1 + R^2 is always positive, the radical always takes real values). For $R > 0$, θ(t) at this fixed solution is negative, so this corresponds to a diving solution. As R increases, the angle of the dive becomes steeper and steeper.

6 Numerical calculations

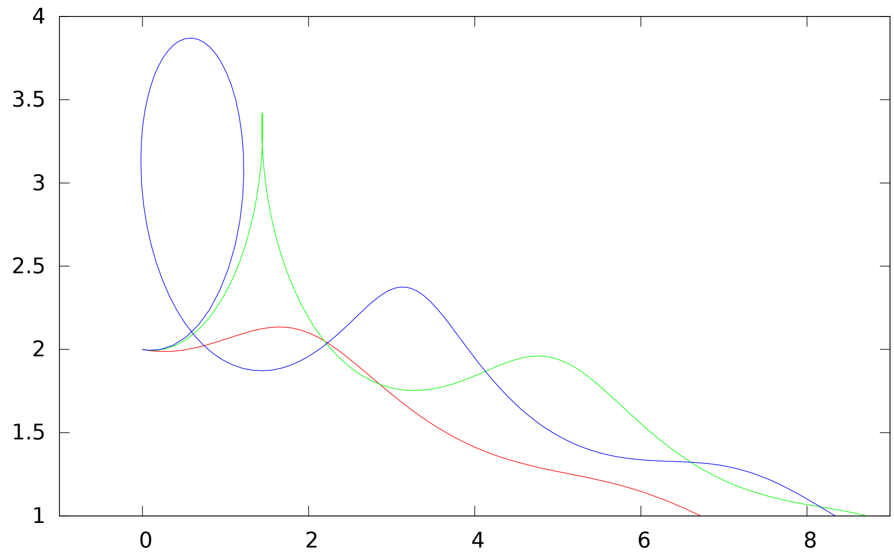

Figure 2: Typical trajectories of a glider.

```
/*
 * The program solves the following system of four
 * first order differential equations, which describe
 * the motion of a glider
 *
 *   v' = -sin(theta) - v^2 / R
 *   theta' = - cos(theta)/v + v
 *   x' = v cos(theta)
 *   y' = v sin(theta)
 *
 * Here v is the dimensionless speed of the glider,
 * theta is the angle that the velocity direction
 * makes with the horizontal, x and y are
 * dimensionless cartesian coordinates of the glider.
 *
 * The step-size of the integrator is automatically
 * adjusted by the controller to maintain the
 * requested accuracy
 */

#include <stdio.h>
```

```
#include <math.h>

#include <gsl/gsl_errno.h>
#include <gsl/gsl_odeiv2.h>

int func (double t, const double y[], double f[],
          void *params)
{
  double R = *(double *) params;

  f[0] = -sin(y[1]) - y[0]*y[0]/R;
  f[1] = -cos(y[1])/y[0] + y[0];
  f[2] = y[0]*cos(y[1]);
  f[3] = y[0]*sin(y[1]);

  return GSL_SUCCESS;
}

int main (void)
{
  size_t neqs = 4;          /* number of equations */
  double eps_abs = 1.e-8,
    eps_rel = 0.;           /* desired precision */
  double stepsize = 1e-6;   /* initial integration step */
  double R = 5.;            /* the aerodynamic efficiency */
  double t = 0., t1 = 120.; /* time interval */
  int status;
  /*
   * Initial conditions
   */
  //double y[4] = { 1.3, -0.1, 0., 2. };
  //double y[4] = { 2.3, -0.1, 0., 2. };
  double y[4] = { 3.3, -0.1, 0., 2. };

  /*
   * Explicit embedded Runge-Kutta-Fehlberg (4,5) method.
   * This method is a good general-purpose integrator.
   */
  gsl_odeiv2_step    *s = gsl_odeiv2_step_alloc
                          (gsl_odeiv2_step_rkf45, neqs);
  gsl_odeiv2_control *c = gsl_odeiv2_control_y_new (eps_abs,
```

```
                                                    eps_rel);
  gsl_odeiv2_evolve  *e = gsl_odeiv2_evolve_alloc (neqs);

  gsl_odeiv2_system sys = {func, NULL, neqs, &R};

  /*
   * Evolution loop
   */
  while ( (t < t1) && (y[3] > 0) )
  {
    status = gsl_odeiv2_evolve_apply (e, c, s, &sys, &t,
                                      t1, &stepsize, y);

    if (status != GSL_SUCCESS) {
      printf ("Troubles: % .5e  % .5e  % .5e  % .5e  % .5e\n",
              t, y[0], y[1], y[2], y[3]);
      break;
    }

    printf ("% .5e  % .5e  % .5e  % .5e  % .5e\n",
            t, y[0], y[1], y[2], y[3]);
  }

  gsl_odeiv2_evolve_free (e);
  gsl_odeiv2_control_free (c);
  gsl_odeiv2_step_free (s);

  return 0;
}
```