Name: ______________________________

Date: ______________________________

Question:	1	2	3	4	Total
Points:	30	20	20	30	100
Score:					

1. A a point mass that can move along a straight line is attached to an end of an ideal elastic spring. (The other end of the spring is fixed.) A viscous friction force proportional to the cube of the velocity is acting on the mass. Therefore, the motion of the particle is described by the following differential equation:

$$\ddot{x} + \mu \dot{x}^3 + x = 0, \tag{1}$$

where μ is a positive dimensionless coefficient of nonlinear friction. (Eq. (1) is written in dimensionless units.)

(a) (20 points) Write a program that solves Eq. (1) numerically for $\mu = 0.2$ and for the following initial conditions.

$$x(0) = 1, \quad \dot{x}(0) = 0. \tag{2}$$

Plot $x(t)$ and $\dot{x}(t)$ for $0 < t < 50$.

Submit a printout of your code, your Makefile, a plot of your program's output.

(b) (10 points) Use the method of averaging to find an approximate solution of Eq. (1) with the initial conditions Eq. (2). Plot on the same graph your approximate solution and the numerical solution you obtained in Part (a). Comment on the agreement between the graphs.

Hints:

1. Search the analytic solution for Eq. (1) in the following form.

$$x = a(t)\cos\left(t + \psi(t)\right), \tag{3}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -a(t)\sin\left(t + \psi(t)\right). \tag{4}$$

2. To calculate averages you may need the following identities.

$$\overline{\sin^4(\phi)} = \frac{3}{8}, \tag{5}$$

$$\overline{\sin^3(\phi)\cos(\phi)} = 0, \tag{6}$$

where overbar denotes averaging over the period.

$$\overline{\ldots} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \ldots \tag{7}$$

2. (20 points) **Symplectic Euler vs "regular" Euler ODE integrators**

   Write two programs that solve Newton's equation for harmonic oscillator,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x = 0, \tag{8}$$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= v, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -x. \end{cases} \tag{9}$$

   with the initial conditions

$$x(0) = 1, \quad \dot{x}(0) = 0. \tag{10}$$

   One of your programs has to solve Eq. (8) using the symplectic Euler algorithm,

$$\begin{cases} v_{n+1} &= v_n - x_n \Delta t, \\ x_{n+1} &= x_n + v_{n+1} \Delta t \end{cases} \tag{11}$$

   The other one has to solve Eq. (8) using the first order Euler algorithm,

$$\begin{cases} v_{n+1} &= v_n - x_n \Delta t, \\ x_{n+1} &= x_n + v_n \Delta t \end{cases} \tag{12}$$

   In both programs use $\Delta t = 0.01$ and follow the evolution of the sytem for $N = 10000$ steps. Plot the phase portraits $v(x)$ produced by the programs. Discuss the difference between the programs' results.

   Do not use arrays to store x_n and v_n; print the results as soon as you calculated them. Note: the style and readability of your code is going to be graded in this assignment.

3. (20 points) **Volume of superellipsoid**

   Use Monte Carlo integration to calculate the volumes, $V(n)$, of the following superellipsoids

$$|x|^n + |y|^n + |z|^n = 1, \quad n = 1, 2, \ldots, 9. \tag{13}$$

   Determine the errors of your calculations. Adjust the number of generated random points such that the errors are less than 0.01. Plot the graph $V(n)$.

   Hint: to check whether your results make sense, note that a regular octahedron is a special case of superellipsoid Eq. (13) for $n = 1$; thus $V(1) = \frac{4}{3}$. For $n = 2$ Eq. (13) is the equation for a unit sphere; thus $V(2) = \frac{4}{3}\pi$. Furthermore, for large n the superellipsoid Eq. (13) is approaching the cube with edge length 2; thus $V(n) \to 8.$ for large n.

   Submit a printout of your code, your Makefile, a graph of your results.

4. **Lagrange points in the restricted three body problem**

   Consider two large gravitating masses, M_1 and M_2 (think about e.g. the Moon and the Earth), $\alpha \equiv \frac{M_2}{M_1} \ll 1$, orbiting their center of mass on circular orbits. Lagrange points are the five positions in an orbital configuration where a small object (think about a satellite) affected only by gravity can be part of a constant-shape pattern with two larger masses.

   (a) (10 points) Analytically find the positions of two Lagrange points, which are **not** located along the line connecting M_1 and M_2.[1] Assume that the smallest object does not affect the motion of two large masses.

   (b) (20 points) Write a program that numerically solves the three-body problem. Check that a small mass placed into the two Lagrange points that you found in Part (a) indeed stays at fixed distance from the two heavy masses. Determine numerically if the Lagrange points are stable or unstable equilibria. Conduct your numerical calculations for $\alpha = 0.0123$.

   Hint: the three body problem of this question is called *circular restricted three body problem*.

[1] The positions of those two Lagrange points have simple analytic expression.