The Three-Body Problem

Adapted from Richard Fitzpatrick, *Newtonian Dynamics*

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1 Introduction

An isolated dynamical system consisting of two moving point masses exerting forces on one another — which is usually referred to as a two-body problem — can always be converted into an equivalent one-body problem. In particular, this implies that we can exactly solve a dynamical system containing two gravitationally interacting point masses, since the equivalent one-body problem is exactly soluble. What about a system containing three gravitationally interacting point masses? Despite hundreds of years of research, no exact solution of this famous problem — which is generally known as the three-body problem — has ever been found. It is, however, possible to make some progress by restricting the problem’s scope.

2 The Circular Restricted Three-Body Problem

Consider a mechanical system consisting of three gravitationally interacting point masses, $M_1$, $M_2$, and $m$. Suppose, that the third mass, $m$, is so much smaller than the other two that it has a negligible effect on their motion. Suppose, further, that the first two masses, $M_1$ and $M_2$, execute a circular orbit about their common center of mass. This simplified problem is known as the circular restricted three-body problem.

Let us further assume, to simplify the presentation of the final calculations, that mass $m$ moves in the plane of the orbital motion of masses $M_1$ and $M_2$.

Let $\omega$ be the constant orbital angular velocity of masses $M_1$ and $M_2$ on the circular orbit. We can find $\omega$ by equating $F_{cp}$, the centripetal force acting upon the mass $\mu = \frac{M_1 M_2}{M_1 + M_2}$ (the equivalent one-body problem), and $F_g$, the force of gravitational attraction between masses $M_1$ and $M_2$:

$$F_{cp} = \frac{M_1 M_2}{M_1 + M_2} \frac{v^2}{R}, \quad F_g = G \frac{M_1 M_2}{R^2},$$  

(1)

where $G$ is the gravitational constant, $v$ is the constant linear velocity of mass $\mu$. From Eq. (1)

$$v^2 = G \frac{M_1 + M_2}{R}. \quad (2)$$  

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On the other hand, the period of orbital motion on a circular orbit, $T$, is

$$ T = \frac{2\pi R}{v}, $$

thus,

$$ \omega \equiv \frac{2\pi}{T} = \frac{v}{R}, \quad \omega^2 = \frac{v^2}{R^2}. $$

Substituting Eq. (2) into Eq. (4), we arrive at the following expression.

$$ \omega^2 = G \frac{M_1 + M_2}{R^3}. $$

Let us define a Cartesian coordinate system $(\xi, \eta, \zeta)$ in an inertial reference frame whose origin coincides with the center of mass, $C$, of the two orbiting masses, $M_1$ and $M_2$. Let the orbital plane of these masses coincide with the $\xi$-$\eta$ plane, and let them both lie on the $\xi$-axis at time $t = 0$ — see Figure 1. Suppose that $R$ is the constant distance between the two orbiting masses, $r_1$ the constant distance between mass $M_1$ and the origin, and $r_2$ the constant distance between mass $M_2$ and the origin.

Let the third mass have position vector $\vec{r} = (\xi, \eta, 0)$. The Cartesian components of the equation of motion of this mass are thus

$$ \ddot{\xi} = -GM_1 \frac{(\xi - \xi_1)}{\rho_1^3} - GM_2 \frac{(\xi - \xi_2)}{\rho_2^3}, $$

$$ \ddot{\eta} = -GM_1 \frac{(\eta - \eta_1)}{\rho_1^3} - GM_2 \frac{(\eta - \eta_2)}{\rho_2^3}. $$
where
\[\rho_1^2 = (\xi - \xi_1)^2 + (\eta - \eta_1)^2, \quad (8)\]
\[\rho_2^2 = (\xi - \xi_2)^2 + (\eta - \eta_2)^2. \quad (9)\]

3 Co-Rotating Frame

Let us transform to a non-inertial frame of reference rotating with angular velocity \(\omega\) about an axis normal to the orbital plane of masses \(M_1\) and \(M_2\), and passing through their center of mass. The masses \(M_1\) and \(M_2\) are stationary in this new reference frame. Let us define a Cartesian coordinate system \((X, Y)\) in the rotating frame of reference which is such that masses \(M_1\) and \(M_2\) always lie on the \(X\)-axis. Let the position vector of mass \(m\) be \(\vec{r} = (x, y)\) — see Figure 2.

The masses \(M_1\) and \(M_2\) have the fixed position vectors
\[\vec{r}_1 = (-\alpha R, 0, 0) \quad \vec{r}_2 = ((1 - \alpha)R, 0, 0) \quad (10)\]
in our new coordinate system. Indeed, by the definition of the center of mass,
\[r_1 M_1 = r_2 M_2. \quad (11)\]
on the other hand,
\[r_1 + r_2 = R. \quad (12)\]
Solving Eqs. (11) and (12), we obtain,
\[r_1 = \frac{M_2}{M_1 + M_2} R, \quad r_2 = \frac{M_1}{M_1 + M_2} R = \left(1 - \frac{M_2}{M_1 + M_2}\right) R, \quad (13)\]
i.e. in Eq. (10)
\[\alpha = \frac{M_2}{M_1 + M_2} \quad (14)\]

The equation of motion of mass \(m\) in the rotating reference frame are obtained by including into Eqs. (6), (7) two additional forces — Coriolis force \(\vec{F}_{\text{cor}}\) and centrifugal force \(\vec{F}_{\text{cf}}\):
\[\vec{F}_{\text{cf}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = m\omega^2 \vec{r}, \quad (15)\]
\[\vec{F}_{\text{cor}} = -2m\vec{\omega} \times \vec{\dot{r}} = 2m\omega (-\hat{x}\dot{y} + \hat{y}\dot{x}), \quad (16)\]
\[\vec{\ddot{r}} = -GM_1 \frac{(\vec{r} - \vec{r}_1)}{\rho_1^3} - GM_2 \frac{(\vec{r} - \vec{r}_2)}{\rho_2^3} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\vec{\omega} \times \vec{\dot{r}}, \quad (17)\]
where \(\vec{\omega} = (0, 0, \omega)\), and
\[\rho_1^2 = (x + \alpha R)^2 + y^2, \quad (18)\]
\[\rho_2^2 = (x - (1 - \alpha)R)^2 + y^2. \quad (19)\]
Here, the last two terms on the right-hand side of Eq. (17) are the centrifugal acceleration and the Coriolis acceleration.

The components of Eq. (17) reduce to

\[
\ddot{x} = -\frac{G M_1 (x + \alpha R)}{\rho_1} - \frac{G M_2 (x - (1 - \alpha)R)}{\rho_2} + \omega^2 x + 2 \omega \dot{y}, \tag{20}
\]

\[
\ddot{y} = -\frac{G M_1}{\rho_1} y - \frac{G M_2 y}{\rho_2} + \omega^2 y - 2 \omega \dot{x}. \tag{21}
\]

4 Jacobi integral

Eqs. (20), (21) can be rewritten as following.

\[
\ddot{x} - 2 \omega \dot{y} = -\frac{\partial U}{\partial x}, \tag{22}
\]

\[
\ddot{y} + 2 \omega \dot{x} = -\frac{\partial U}{\partial y}. \tag{23}
\]

where

\[
U = -\frac{G M_1}{\rho_1} - \frac{G M_2}{\rho_2} - \frac{\omega^2}{2} (x^2 + y^2) \tag{24}
\]

is the sum of the gravitational and centrifugal potentials.
Now, it follows from Eqs (22)–(23) that
\[ \ddot{x} \dot{x} - 2 \omega \dot{x} \dot{y} = -\dot{x} \frac{\partial U}{\partial x}, \]  
(25) 
\[ \ddot{y} \dot{y} + 2 \omega \dot{x} \dot{y} = -\dot{y} \frac{\partial U}{\partial y}. \]  
(26)

Summing the above equations, we obtain
\[ \frac{d}{dt} \left[ \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + U \right] = 0. \] 
(27)

In other words,
\[ C = -2U - v^2 \]  
(28)
is a constant of the motion, where \( v^2 = \dot{x}^2 + \dot{y}^2 \). \( C \) is called the Jacobi integral. The mass \( m \) is restricted to regions in which
\[ -2U \geq C, \] 
(29)
since \( v^2 \) is a positive definite quantity.

## 5 Dimensionless form of the equations

No analytic solutions of Eqs. (20)–(21) are known. Our goal is to solve them numerically. As the first required step, we convert the to a dimensionless form.

Circular restricted three body problem has two natural scales: the distance, \( R \), between masses \( M_1 \) and \( M_2 \), and the characteristic time of their orbital motion \( 1/\omega \). Let us introduce dimensionless variables by measuring the coordinates \( x \) and \( y \) in units of \( R \), thus introducing new unknowns \( u \) and \( v \) as following,
\[ u \equiv \frac{x}{R}, \quad v \equiv \frac{y}{R}, \] 
(30)

Let us measure time \( t \) in units of \( 1/\omega \), introducing dimensionless variable \( \tau \),
\[ \tau \equiv \omega t. \] 
(31)

“Old” derivatives with respect to time are going to have the following forms:
\[ \dot{x} \equiv \frac{dx}{dt} = \frac{d(u R)}{d(\tau/\omega)} = \omega R \frac{du}{d\tau}, \] 
(32)
\[ \ddot{x} \equiv \frac{d}{dt} \left( \frac{dx}{dt} \right) = \frac{d}{d\tau} \left( \omega R \frac{du}{d\tau} \right) = \omega R \frac{d}{d\tau/\omega} \left( \frac{du}{d\tau} \right) = \omega^2 R \frac{d^2 u}{d\tau^2}. \] 
(33)

Similarly,
\[ \dot{y} = \omega R \frac{dv}{d\tau} \] 
(34)
\[ \ddot{y} = \omega^2 R \frac{d^2 v}{d \tau^2} \]  

Substituting Eqs. (32)–(35) into Eqs. (20), (21), we get:

\[
\begin{align*}
\omega^2 R \frac{d^2 u}{d \tau^2} &= -\frac{GM_1 R (u + \alpha)}{\rho_1^3} - \frac{GM_2 R (u - 1 + \alpha)}{\rho_2^3} + \omega^2 R u + 2 \omega^2 R \frac{dv}{d \tau}, \\
\omega^2 R \frac{d^2 v}{d \tau^2} &= -\frac{GM_1 R v}{\rho_1^3} - \frac{GM_2 R v}{\rho_2^3} + \omega^2 R v - 2 \omega^2 R \frac{du}{d \tau}.
\end{align*}
\]

Here \( \rho_1 \) and \( \rho_2 \) expressed via dimensionless parameters are as following:

\[
\begin{align*}
\rho_1 &= R \left( (u + \alpha)^2 + v^2 \right)^{\frac{1}{2}} = Rd_1, \\
\rho_2 &= R \left( (u - 1 + \alpha)^2 + v^2 \right)^{\frac{1}{2}} = Rd_2,
\end{align*}
\]

where

\[
\begin{align*}
d_1 &\equiv \left( (u + \alpha)^2 + v^2 \right)^{\frac{1}{2}}, \\
d_2 &\equiv \left( (u - 1 + \alpha)^2 + v^2 \right)^{\frac{1}{2}}.
\end{align*}
\]

Dividing each term in Eqs. (36)–(37) by \( \omega^2 R \), we arrive at the following equations:

\[
\begin{align*}
\frac{d^2 u}{d \tau^2} &= -\frac{GM_1}{\omega^2 R^3} \frac{(u + \alpha)}{d_1^3} - \frac{GM_2}{\omega^2 R^3} \frac{(u - 1 + \alpha)}{d_2^3} + u + 2 \frac{dv}{d \tau}, \\
\frac{d^2 v}{d \tau^2} &= -\frac{GM_1}{\omega^2 R^3} \frac{v}{d_1^3} - \frac{GM_2}{\omega^2 R^3} \frac{v}{d_2^3} + v - 2 \frac{du}{d \tau}.
\end{align*}
\]

Noticing that

\[
\frac{GM_1}{\omega^2 R^3} = \frac{M_1}{M_1 + M_2} \equiv 1 - \alpha
\]

and

\[
\frac{GM_2}{\omega^2 R^3} = \frac{M_2}{M_1 + M_2} \equiv \alpha
\]

we arrive at the following equations.

\[
\begin{align*}
\frac{d^2 u}{d \tau^2} &= -(1 - \alpha) \frac{(u + \alpha)}{d_1^3} - \alpha \frac{(u - 1 + \alpha)}{d_2^3} + u + 2 \frac{dv}{d \tau}, \\
\frac{d^2 v}{d \tau^2} &= -(1 - \alpha) \frac{v}{d_1^3} - \alpha \frac{v}{d_2^3} + v - 2 \frac{du}{d \tau}.
\end{align*}
\]

Equations (46)–(47) can be rewritten in a compact form

\[
\begin{align*}
\ddot{u} &= -\frac{\partial U}{\partial v} + 2 \dot{v}, \\
\ddot{v} &= -\frac{\partial U}{\partial v} - 2 \dot{u},
\end{align*}
\]
where

$$U(u, v) = -\frac{1 - \alpha}{d_1} - \frac{\alpha}{d_2} - \frac{1}{2} \left(u^2 + v^2\right)$$  \hspace{2cm} (50)$$

is the dimensionless version of Eq. (24).

Equations (46)- (47) are dimensionless and contain a single parameter, $\alpha$. Some of the results of their numerical solution are presented in Figs. 3 and 4. A fragment of the code used for calculations is presented in the Appendix A.

Figure 3: Arenstorf periodic orbits for $\alpha = 0.012277471$ and initial conditions $x(0) = 0.994$, $y(0) = 0$, $\dot{x}(0) = 0$; left subfigure: $\dot{y}(0) = -2.0317326295573368357302057924$, right subfigure: $\dot{y}(0) = -2.00158510637908252240537862224$.

Figure 4: Chaotic orbit: $\alpha = 0.5$, $x(0) = 1$, $y(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = 0$. 
Appendix A

A fragment of a C code to solve the restricted three-body problem using gsl library.

```c
int func (double t, const double yy[], double f[], void *params)
{
    double a = *(double *) params;
    double d1, d2;
    double x = yy[0], y = yy[1], vx = yy[2], vy = yy[3];

    d1 = pow((x + a)*(x + a) + y*y, 1.5);
    d2 = pow((x + a - 1.)*(x + a - 1.) + y*y, 1.5);

    f[0] = vx;
    f[1] = vy;
    f[2] = -(1. - a)*(x + a)/d1 - a*(x + a - 1.)/d2 + x + 2*vy,
    f[3] = -(1. - a)*y/d1 - a*y/d2 + y - 2*vx;

    return GSL_SUCCESS;
}
```