Name: ______________________________

Date: ______________________________

Question:	1	2	3	Total
Points:	25	15	50	90
Score:				

1. Imagine that as a part of the solution of a particular physical problem you need repeated calculations of the following expression:
$$\frac{1-\cos(x)}{\sin^2(x)}. \tag{1}$$
for small x, $10^{-11} < x < 10^{-1}$.

   (a) (5 points) Analytically calculate the limit
   $$\lim_{x\to 0}\frac{1-\cos(x)}{\sin^2(x)}.$$

   (b) (10 points) Write a small C program (15–20 lines, not counting comments) that evaluates and prints expression Eq. (1) for $x = 0.1, 0.01, \ldots, 0.00000000001$. Compare the results with your analytic calculation. Why there is a difference?

   (c) (10 points) Rewrite Eq. (1) into a form that is suitable for numerical calculations. Modify your program to print (with 10 figures after the decimal point) side by side x, the result given by Eq. (1), and the result of your modified expression.

   Submit the printout of your code, your Makefile, final program output, explanation of the result in b. (Note: the style of your code is going to be graded in this assignment.)

2. Multiplication algorithm for large integers

   (a) Convince yourself that traditional grade-school addition and multiplication techniques of two n-bit integers are $O(n)$ and $O(n^2)$ operations respectedly.

   (b) Suppose x and y are two n-bit integers (think of n in the range $2048 - 4096$), and assume for convenience that n is a power of 2. As a first step toward multiplying x and y, split each of them into their left and right halves, which are $n/2$ bits long:
   $$x = 2^{n/2}x_L + x_R, \tag{2}$$
   $$y = 2^{n/2}y_L + y_R. \tag{3}$$
   For instance, if $x = 10110110_2$ (the subscript 2 means binary) then $x_L = 1011_2$, $x_R = 0110_2$, and $x = 2^4 \times 1011_2 + 0110_2$. The product of x and y can then be rewritten as
   $$xy = (2^{n/2}x_L + x_R)(2^{n/2}y_L + y_R) = 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R. \tag{4}$$

Let's compute xy via the expression on the right of Eq. (4). The additions take linear time, as do the multiplications by powers of 2 (which are merely left-shifts). The significant operations are the four n/2-bit multiplications, $x_L y_L$, $x_L y_R$, $x_R y_L$, $x_R y_R$; these we can handle by four recursive calls.

Thus our method for multiplying n-bit numbers starts by making recursive calls to multiply these four pairs of $n/2$-bit numbers (four subproblems of half the size), and then evaluates the preceding expression in $O(n)$ time. Writing $T(n)$ for the overall running time on n-bit inputs, we get the recurrence relation

$$T(n) = 4T(n/2) + O(n). \tag{5}$$

(c) (5 points) Show that the running time of the algorithm Eq. (4) is $O(n^2)$.

(d) This is the same running time as the traditional grade-school multiplication technique. So we have a radically new algorithm, but we havent yet made any progress in efficiency.

(e) Although the expression for xy Eq. (4) seems to demand four $n/2$-bit multiplications, three will do: $x_L y_L$, $x_R y_R$, and $(x_L + x_R)(y_L + y_R)$, since

$$x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) x_L y_L x_R y_R. \tag{6}$$

The resulting algorithm has an improved running time of

$$T(n) = 3T(n/2) + O(n). \tag{7}$$

The point is that now the constant factor improvement, from 4 to 3, occurs at every level of the recursion, and this compounding effect leads to a dramatically lower time bound.

(f) (10 points) Find the running time of the multiplication algorithm Eq. (6), (7). Estimate of the speedup for multiplication of two 4096 bit integers.

3. A mechanical system consists of a rigid pendulum (a massless rigid rod of length l and a point mass at its end) whose suspension point is vibrating with frequency ω and amplitude a along the vertical direction. The equation of motion describing such a system is as following.

$$\ddot{\phi} + \omega_0^2 \left(1 - \frac{a}{l}\frac{\omega^2}{\omega_0^2}\sin(\omega t)\right)\sin(\phi) = 0. \tag{8}$$

Here ϕ is the angle between the pendulum's rod and the vertical "down" direction and $\omega_0 = \sqrt{\frac{l}{g}}$ is the frequency of pendulum's natural oscillations.

An analysis of Eq. (8) predicts that for sufficient large value of ω and a vertical "up" position ($\phi = \pi$) of the pendulum is stable. Your task is to check this numerically.

(a) (10 points) Rewrite Eq. (8) in the dimensionless form.

$$\ddot{\phi} + \left(1 - \frac{a}{l}\Omega^2 \sin(\Omega\tau)\right)\sin(\phi) = 0, \tag{9}$$

where

$$\Omega = \frac{\omega}{\omega_0}.$$

Hint: introduce dimensionless time-like variable $\tau = \omega_0 t$.

(b) (20 points) Write a program that solves Eq. (9) numerically for the following fixed values of parameters,

$$\frac{a}{l} = 0.01$$

and the following initial conditions.

$$\phi(0) = 0.99\pi.$$

(c) (10 points) To see the stable and the unstable oscillations around the vertical "up" direction, conduct your calculations for several values of Ω between 100 and 200.

(d) (10 points) Determine by trial and error (with the relative error $\sim 5\%$) the critical value of Ω when the vertical up pendulum position looses its stability.