Date:

Name:

Show all your work and indicate your reasoning in order to receive the most credit.

1. (60pt) A classical model in mathematical ecology is the Lotka-Volterra predator-prey model. Consider a simple ecosystem consisting of rabbits that have an infinite supply of food and foxes that prey on the rabbits for their food. This is modeled by a pair of nonlinear, 1st-order differential equations:

$$\frac{dr}{dt} = 2r - \alpha r f, \quad r(0) = r_0, \tag{1}$$

$$\frac{df}{dt} = -f + \alpha r f, \quad f(0) = f_0, \tag{2}$$

where t is time, $r(t)$ is the number of rabbits, $f(t)$ is the number of foxes, and α is a positive constant. If $\alpha = 0$, the two populations do not interact, the rabbits multiply, and the foxes die off from starvation. If $\alpha > 0$, the foxes encounter the rabbits with a probability that is proportional to the product of their numbers. Such an encounter results in a decrease in the number of rabbits and (for less obvious reasons) an increase in the number of foxes.

The solutions to this nonlinear system cannot be expressed in terms of other known functions; the equations must be solved numerically. It turns out that the solutions are always periodic, with a period that depends on the initial conditions. In other words, for any $r(0)$ and $f(0)$, there is a value $t = t_p$ when both populations return to their original values. Consequently, for all t, $r(t + t_p) = r(t)$, $f(t + t_p) = f(t)$:

   (a) Compute the solution with $r_0 = 300$, $f_0 = 150$, and $\alpha = 0.01$. You should find that t_p is close to 5. Make two plots, one of r and f as functions of t and one a phase plane plot with r as one axis and f as the other.

   (b) The point

   $$(r_0, f_0) = (\frac{1}{\alpha}, \frac{2}{\alpha})$$

   is a stable equilibrium point. If the populations have these initial values, they do not change. Numerically check this statement for $\alpha = \frac{1}{100}$.

For (a) and (b) use an integration algorithm that doesn't require Jacobian.

Provide the printouts of your (formatted) C code, your graphs, your makefile, and gnuplot script you used to plot the graphs.

2. (40pt) Two-dimensional flame propagation (think about a match squeezed between two parallel slabs of non-flammable material) can be described by the following ordinary differential equation:

$$\frac{dr}{dt} = r - r^2, \quad r(0) = \delta, \tag{3}$$

where the scalar (dimensionless) variable $r(t)$ represents the radius of the fire disk vs (dimensionless) time.

When you light a match, the disk of flame grows rapidly until it reaches a critical size. Then it remains at that size because the amount of oxygen being consumed by the combustion in the interior of the disk balances the amount available through the interface.

The r and r^2 terms in Eq. (3) come from the interface area and the volume of the flame. The only parameter is the initial radius, δ, which is "small".

Integrate Eq. (3) numerically using an algorithm that doesn't require Jacobian. Chose $\delta = 0.0001$ and follow the evolution of the flame for $0 \leq t \leq 20$. Plot the graph $r(t)$.

Provide the printouts of your (formatted) C code, your makefile, and the gnuplot script you used to produce the graph.