A Metric Approach to Transformation Optics

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Abstract

The conventional method for calculating the dielectric tensor for metamaterial relies on finding a transformation from the physical space containing the medium to a virtual electromagnetic space. Such a transformation exists if, and only if, the space is flat, which greatly restricts the way in which we are able to manipulate light. Furthermore, the transformation medium must not alter the temporal coordinate. Here, we show an alternate method for calculating the dielectric tensor that stems from the light ray trajectory equation derived from the metric. Thus, the process of defining an explicit transformation is circumvented entirely. We show that this method reproduces the correct results for the cylindrical cloaking device, and we use it to derive the material parameters of a device that gives rise to a curved space in which light rays follow closed Keplerian orbits.

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I. INTRODUCTION

In recent years, dielectric media has been engineered with remarkable optical properties derived from the geometry and structure of the constituent unit cells [1]. These materials, called metamaterials, generally consist of inductive and capacitive circuit elements with prescribed electric and magnetic resonances. The subwavelength components are assembled in arrays such that the incident electromagnetic radiation sees the medium as being continuous and described by macroscopic material parameters: electric permittivity ϵ, and magnetic permeability μ. The tunability of these devices yields tremendous control over the material parameters, and gives rise to optical phenomena like negative refraction [2, 3] and perfect imaging [4, 5], among others.

Moreover, metamaterials can be used to manipulate light in novel ways by placing the electromagnetic fields in desired locations. This is equivalent to changing the geometry or manifold on which the fields traverse. Since Maxwell's equations are form invariant, the coordinate transformation that distorts the space can be applied to the permittivity and permeability tensors to effectively map electromagnetism onto a new space. This is the procedure of transformation optics [6], whereby metamaterial induces a coordinate transformation from physical space containing the material, to a new electromagnetic space. In this way, the material creates a new geometry and information about this coordinate transformation is essentially embedded in the material properties, ϵ and μ, of the transformation medium. It has been known for quite some time that material parameters may appear as geometry in Maxwell's equations [7–9], but this particular field of optics has burgeoned in the past few years as researchers have conceived of materials that can rotate fields [10], concentrate and expand fields [11], and cloak objects [12, 13].

However, this method suffers from two limitations. Firstly, transformations from Cartesian coordinates to the desired coordinate system are only possible for flat spaces described by a vanishing Riemann curvature tensor

$$R^{\rho}{}_{\alpha\beta\gamma} = 0, \tag{1}$$

and therefore conventional transformation media restricts us to flat geometries. We can relax this restriction by developing a new approach that accommodates curved spacetimes, thereby affording us more flexibility in manipulating light. While some researchers have recently proposed ways of dealing with non-Euclidean spaces that allow for broadband invisibility [14], alternate approaches might include embedding techniques similar to those found in general relativity in addition to conformal transformations, such that the new (perhaps higher dimensional) space is flat or conformally so. However, in this paper, we shall use an effective Fermat's principle formulation in addition to the Plebanski constituent equations [9] to express the material parameters in terms of the geometry [15].

The second limitation stems from the procedure commonly found in the literature, where the first step is to conceive of a transformation that will give rise to the desired trajectory for a light ray. After this is done, it is simply a matter of seeing how the material properties change for the given transformation. For many systems, such as the spherical cloaking device, the necessary transformation is relatively easy to intuit and the procedure is straightforward. However, one can imagine a more complicated system for which the transformation is not obvious a priori.

Instead, what is known a priori, is the equation of motion for the light ray; it describes the way in which we wish to manipulate light. Thus, we introduce a method here that bypasses the need for intuiting a transformation by using the equation of motion to calculate the metric components of the new space. If the metric is such that the Riemann tensor vanishes, then the space is flat and the material properties can be calculated in the usual manner by the following relations [13]

$$\begin{aligned} \epsilon^{i'j'} &= \left| det\left(g^{i'j'}\right)\right|^{-1/2} g^{i'j'}\epsilon, \\ \mu^{i'j'} &= \left| det\left(g^{i'j'}\right)\right|^{-1/2} g^{i'j'}\mu, \end{aligned} \tag{2}$$

where Latin indices, i and j, run over the three spatial coordinates and primes denote the transformed space such that the metric in the new space is given by

$$g^{i'j'} = \Lambda^{i'}{}_{k}\Lambda^{j'}{}_{l}g^{kl}. \tag{3}$$

$\Lambda^{i'}{}_{k} = \partial x^{i'}/\partial x^{k}$ is the Jacobian transformation matrix element. However, if the metric describes a curved space, other methods must be used to calculate the material properties, as we will show in this paper. Either way, if the effect of curvature is small compared to the wavelength of the incident radiation, we may use the formulation of geometric optics to calculate the trajectory of the light ray. This condition can be expressed in terms of the Ricci curvature scalar, $\mathscr{R} = R^{\alpha}{}_{\alpha} = g^{\alpha\beta}R_{\alpha\beta}$, via the following equation

$$|\mathscr{R}| \ll \frac{1}{\lambda^2}. \tag{4}$$

(Note that throughout this paper, Greek indices shall begin at zero and run over all coordinates.) The problem is subsequently reduced to finding the metric that gives rise to the correct null geodesic in spacetime. While much of the literature related to transformation optics uses the language of general relativity to describe the geometry of electromagnetism [6], the coupling between the method presented here and general relativity is particularly strong, and we shall work in a $(2, 1)$ spacetime with a negative sign corresponding to the temporal coordinate, and a positive sign corresponding to spatial coordinates.

In Sec. II we demonstrate this method by first showing that the equations of motion for light rays traversing through a cylindrical cloak can be used to obtain the correct transformation and subsequently, the correct material parameters. We then use this technique to derive the metric that allows for Keplerian orbits of light rays within some metamaterial in Sec. III. In Sec. IV we implement isotropic coordinates to calculate the refractive index via Fermat's Principle for the aforementioned material, and finally show that the same result can be achieved by making use of the covariant nature of Maxwell's equations in Sec. V.

II. APPLICATION TO 2D CYLINDRICAL CLOAK

We will first demonstrate the validity of this technique by applying it to a device that is well documented in the literature: the cylindrical cloak first envisaged in [13], and for simplicity, we shall specifically examine the two dimensional cloak. This linear dielectric device is cylindrically anisotropic and inhomogeneous, and is comprised of a shell of inner radius a and outer radius b. The effect of the material is to guide light rays smoothly around a circular region of radius a, so that they emerge from the device on the paths they would have traversed sans cloak. With minimal absorption and scattering, the device renders a circular region that is invisible to observers. We will show a way to calculate the material properties without intuiting a transformation, and we begin by briefly describing a method for obtaining the equation of motion for a light ray traversing through this device, which closely parallels [17]. This equation of motion will be used as a constraint on the metric created by the device, and will ultimately lead to the same transformation contained in [13]. The analysis begins by considering a light ray in vacuum whose trajectory (in polar coordinates) is a straight line given by

$$r = \frac{\rho}{\sin\theta}, \tag{5}$$

where ρ is the impact parameter and r is the distance from the origin to the light ray at a given angle, as shown in Fig. 1(a). Equation (5) describes the trajectory of a light ray outside the metamaterial. While inside the cloaking device, the light ray should bend around a region in space so as to exit the material on the same path that it would have taken in vacuum. To achieve this motion, we place the metamaterial at the origin and modify equation (5) to read

$$r = \frac{\rho}{\sin\theta} + a, \tag{6}$$

so that all rays avoid the cloaked region entirely. Note that the impact parameter for a light ray incident on the outer surface of the device ($r = b$) is given by $\rho = b\sin\theta_i$, where θ_i is the incident angle of the light ray as shown in Fig. 1(b). Thus, (6) becomes

$$r = \frac{b\sin\theta_i}{\sin\theta} + a. \tag{7}$$

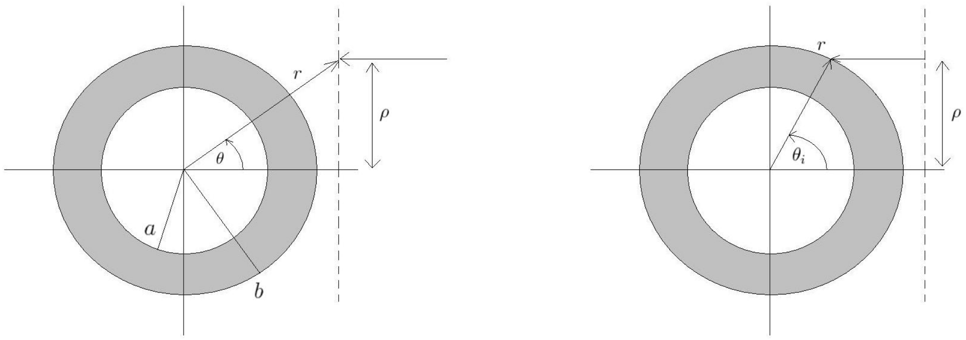
The crucial step is to recognize that (7) is inadequate because it suggests that $r = b + a$ when $\theta = \theta_i$, which is clearly incorrect. The light ray impinging on the outer surface should be a distance $r = b$ from the origin for any incident angle. Thus, the correct version of (7) must contain an additive term, $r'(\theta)$ such that $r(\theta_i) = b$:

$$r(\theta) = \frac{b\sin\theta_i}{\sin\theta} + a + r'(\theta), \tag{8}$$

where $r'(\theta_i) = -a$. We now examine (7) when $\theta = \pi/2$:

$$r = b\sin\theta_i + a. \tag{9}$$

This is too large by an amount exactly equal to the impact parameter of a fictitious light ray impinging on the *inner* surface with the *same* incident angle. Thus, the correction term for this particular angle is given by $r'(\pi/2) = -a\sin\theta_i$, and we see that the magnitude of $r'(\theta)$



(a) (b)

FIG. 1: Trajectory of incident light ray with impact parameter $\rho = r\sin\theta$ (a) approaching a device of inner radius a and outer radius b and (b) impinging on the outer surface of said device such that $\rho = r\sin\theta = b\sin\theta_i$.

must decrease as θ increases to $\pi/2$, after which it begins to increase. Continuing along this vein, we conclude that $r'(\theta) = -a\sin\theta_i/\sin\theta$, so that the correct ray trajectory equation is

$$\begin{aligned} r(\theta) &= \frac{b\sin\theta_i}{\sin\theta} + a - \frac{a\sin\theta_i}{\sin\theta}, \\ &= \frac{(b-a)\sin\theta_i}{\sin\theta} + a. \end{aligned} \tag{10}$$

This is the ray equation for *any* light ray *inside* the cloaking device. However, it should be noted that there is a singularity for the on axis ray ($\theta_i = 0$), for it must be severely deflected as it tries to traverse the device in a perfect circle of radius a. However, the ambiguity associated with the direction in which this ray bends (above or below the device) implies that it is not deflected at all [17, 18]. Differentiating (10) with respect to θ yields

$$\begin{aligned} \frac{dr}{d\theta} &= -\frac{(b-a)\sin\theta_i}{\sin^2\theta}\cos\theta, \\ &= -(r-a)^2\left[\alpha^2 - \frac{1}{(r-a)^2}\right]^{1/2}, \end{aligned} \tag{11}$$

where $\alpha = ((b-a)\sin\theta_i)^{-1}$. Henceforth it is assumed that (11) is known a priori, and we are free to use it as an external constraint in our analysis. We now postulate that the metamaterial will give rise to a metric of the form

$$ds^2 = -\Phi(r)dt^2 + \Psi(r)dr^2 + \chi(r)r^2 d\theta^2. \tag{12}$$

The Lagrangian describing this geometry is given by [19]

$$\mathscr{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\left(-\Phi(r)\dot{t}^2 + \Psi(r)\dot{r}^2 + \chi(r)r^2\dot{\theta}^2\right). \tag{13}$$

Using (13) in the Euler-Lagrange equations [20]

$$\frac{d}{dq}\frac{\partial \mathscr{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathscr{L}}{\partial \dot{x}^\mu} = 0, \tag{14}$$

yields the following equations of motion for the t and θ coordinates respectively

$$\begin{aligned} \Phi\dot{t} &= const. = \ell, \\ \chi r^2\dot{\theta} &= const. = h. \end{aligned} \tag{15}$$

We can now use these relations in the line element itself to derive the equation of motion for the r coordinate as follows [19]

$$\begin{aligned} 0 &= -\Phi\left(\frac{dt}{dq}\right)^2 + \Psi\left(\frac{dr}{dq}\right)^2 + \chi r^2\left(\frac{d\theta}{dq}\right)^2, \\ &= -\Phi\left(\frac{\ell}{\Phi}\right)^2 + \Psi\dot{r}^2 + \chi r^2\left(\frac{h}{\chi r^2}\right)^2, \\ &= -\frac{\ell^2}{\Phi} + \Psi\dot{r}^2 + \frac{h^2}{\chi r^2}. \end{aligned} \tag{16}$$

Changing the differentiation variable to θ yields

$$0 = -\ell^2 + \Psi\Phi\frac{h^2}{\chi^2 r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{h^2\Phi}{\chi r^2}, \tag{17}$$

so that the ray trajectory equation is given by

$$\frac{dr}{d\theta} = \pm\frac{\chi r^2}{h\sqrt{\Phi\Psi}}\left[\ell^2 - \frac{h^2\Phi}{\chi r^2}\right]^{1/2}. \tag{18}$$

By comparing (18) with (11), we see that the negative root is the appropriate one, and when $\alpha = (b/(b-a))(\ell/h)$, we find that:

$$\Phi(r) = 1, \qquad \Psi(r) = \left(\frac{b}{b-a}\right)^2, \qquad \chi(r) = \left(\frac{b}{b-a}\right)^2\left(\frac{r-a}{r}\right)^2. \tag{19}$$

Thus, the line element (12) now reads

$$ds^2 = -dt^2 + \left(\frac{b}{b-a}\right)^2 dr^2 + \left(\frac{b}{b-a}\right)^2 (r-a)^2\, d\theta^2. \tag{20}$$

Equation (20) can be obtained from the line element describing flat space in polar coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 \tag{21}$$

via the following transformation

$$r \to (r-a)\left(\frac{b}{b-a}\right), \tag{22}$$

with inverse transformation given by

$$r \to \frac{b-a}{b} r + a. \tag{23}$$

This is precisely the transformation used in [13] to derive the permittivity tensor via (2). Thus, we have shown that one can essentially reverse the procedure that is commonly found in the literature to calculate the material properties of the metamaterial, thereby eradicating the need for the transformation in the first place.

III. KEPLERIAN ORBITS

Now we will demonstrate the power of this method by using an intuitive equation of motion to derive an otherwise unintuitive metric. More specifically, we would like to obtain a metric induced by metamaterial that will give rise to closed Keplerian orbits for light rays [22]. We begin by considering a cylindrical metamaterial with radius a, and we postulate that it will give rise to the following stationary, cylindrically symmetric line element for $r \leq a$ in the $z = 0$ plane

$$ds^2 = -\Phi(r)dt^2 + \Psi(r)dr^2 + \chi(r)r^2 d\theta^2. \tag{24}$$

The metric for $r > a$ is simply that for Minkowski spacetime. Note that under the following transformation

$$\tilde{r} = \sqrt{\chi} r, \tag{25}$$

the line element becomes

$$\begin{aligned} ds^2 &= -\Phi dt^2 + \frac{\Psi}{\chi}\left(1 + \frac{r}{2\chi}\frac{d\chi}{dr}\right)^{-2} d\tilde{r}^2 + \chi\left(\frac{\tilde{r}^2}{\chi}\right) d\theta^2, &(26)\\ &= -\Phi(r)dt^2 + \tilde{\Psi}(r)d\tilde{r}^2 + \tilde{r}^2 d\theta^2, &(27)\end{aligned}$$

so that for a particular choice of the radial coordinate, the multiplicative function in the metric corresponding to the θ coordinate vanishes [19]. The Lagrangian derived from (24) is given by

$$\mathscr{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\left(-\Phi(r)\dot{t}^2 + \Psi(r)\dot{r}^2 + r^2\dot{\theta}^2\right), \tag{28}$$

where we have dropped tildes for aesthetic reasons. Proceeding as before, we find that

$$0 = -\ell^2 + \Psi\Phi\frac{h^2}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{h^2\Phi}{r^2}. \tag{29}$$

Rather than obtain the differential equation of motion as in the previous section, we separate the equation to obtain the following integral:

$$\theta = \pm \int \frac{\Psi^{1/2} dr}{r^2\left(\frac{\ell^2}{h^2\Phi} - \frac{1}{r^2}\right)^{1/2}}, \tag{30}$$

or

$$\theta = \mp \int \frac{dx}{\sqrt{\frac{\ell^2}{h^2\Psi\Phi} - \frac{x^2}{\Psi}}}, \tag{31}$$

where $x = 1/r$. Equation (31) bears a striking resemblance to the standard integral

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}}, \tag{32}$$

with a solution given by [23]

$$\frac{1}{\sqrt{-\gamma}} \cos^{-1}\left[-\frac{\beta + 2\gamma x}{\sqrt{\beta^2 - 4\alpha\gamma}}\right]. \tag{33}$$

It is well known from the Kepler problem of planetary motion that (33) leads directly to orbits that are conic sections described by the following equation (with $\gamma = -1$):

$$x = R^{-1}\left(1 + e\cos\theta\right), \tag{34}$$

where $R = 2/\beta$ is the radius of circular orbit and $e = \sqrt{1 + 4\alpha/\beta^2}$ is the eccentricity of orbit. By comparing (31) with (32), we find that the following equation must hold in order to achieve similar Keplerian orbits for light rays inside the metamaterial

$$\alpha + \beta x - x^2 = \left(\frac{\ell}{h}\right)^2 \frac{1}{\Psi\Phi} - \frac{x^2}{\Psi}. \tag{35}$$

Due to the arbitrary nature of Φ and Ψ, we are free to impose the constraint that $\Phi = \Psi^{-1}$ for simplicity. Solving for $\Phi(x)$ in the equation above yields

$$\Phi(x) = \left(J^2 - \alpha\right)\frac{1}{x^2} - \frac{\beta}{x} + 1, \tag{36}$$

where $J = \ell/h$. Since the metric must converge to that of Minkowski spacetime at the boundary of the metamaterial, we impose the condition that $\Phi(x = 1/a) = 1$, which implies that $\alpha = J^2 - \beta/a$. Thus the final solution is

$$\Phi(r) = \frac{2r^2 - 2ar + aR}{aR}, \tag{37}$$

so that the line element reads

$$ds^2 = -\left(\frac{2r^2 - 2ar + aR}{aR}\right)dt^2 + \left(\frac{2r^2 - 2ar + aR}{aR}\right)^{-1} dr^2 + r^2 d\theta^2, \tag{38}$$

for $r \leq a$. This metric will give rise to closed Keplerian orbits inside the metamaterial if the radius and/or apoapsis distance is less than the radius of the metamaterial. For bounded orbit, the eccentricity must be between zero and one (inclusive) which implies that

$$-1 \leq \frac{4\alpha}{\beta^2} \leq 0. \tag{39}$$

Using the expression for α and the fact that $\beta = 2/R$ in the equation above yields

$$-1 \leq \frac{aR^2J^2 - 2R}{a} \leq 0. \tag{40}$$

Thus, the incident light ray must have angular momentum subject to the following constraint

$$\frac{2R-a}{aR^2} \le J^2 \le \frac{2}{aR}. \tag{41}$$

In this way, one first specifies the physical radius of the device a, as well as the desired radius of a circular orbit $R \le a$, and tunes the device accordingly. Then, the angular momentum of the light ray (a constant of motion) has to be chosen in terms of a and R to yield the desired eccentricity, in accordance with (41).

It is important to note that this metric (38) describes a *curved* spacetime in which the Riemann tensor does not identically vanish. Thus, it is impossible to find a transformation that will distort a flat Euclidean metric of electromagnetic space to the physical space described by (38). Moreover, the space does not contain constant curvature and therefore cannot be expressed as globally conformal to flat space [24]. This implies that such a curved space *cannot* be created by conventional transformation media. However, it is possible to describe the curved geometry by the material parameters of some dielectric, as we will later show in this paper.

It should also be noted that the Ricci scalar for the metric described by (38) is given by

$$\mathscr{R} = \frac{4(a-3r)}{aRr}, \tag{42}$$

and diverges for small r values. Thus, the Ricci scalar does not satisfy (4) for the entire domain of r, and therefore the geometric ray formulation is only valid for $a/3 \le r \le a$.

IV. ISOTROPIC COORDINATES

It is both instructive and advantageous to express the line element (38) in terms of isotropic coordinates before calculating the dielectric tensors, so that it becomes *spatially* conformal to flat space. Though spacetime distances are measured differently between conformal spaces, angles are preserved in the conformal pair, as are their null geodesics [6, 19, 24]. It should be noted that not all metrics lend themselves to conformal form, though this particular one does. For the following transformation

$$\rho = \frac{a(2R-a)r}{a(R-r)+\sqrt{(aR)(aR-2ar+2r^2)}}, \tag{43}$$

the line element takes the form:

$$ds^2 = \frac{1}{P^2(\rho)} \left[- \left(\rho^2 - a(a-2R)\right)^2 dt^2 + (2aR)^2 \left(d\rho^2 + \rho^2 d\theta^2\right) \right], \tag{44}$$

where $P(\rho) = -\rho^2 + 2a\rho - a(a-2R)$. Note that $r = a$ implies that $\rho = a$, and the line element converges to that of Minkowski spacetime. When $R = a$, we have

$$ds^2 = -\Sigma^2(\rho)dt^2 + \Omega^2(\rho)\left(d\rho^2 + \rho^2 d\theta^2\right), \tag{45}$$

where $\Omega(\rho) = 2a^2/(a^2 + 2a\rho - \rho^2)$ and $\Sigma(\rho) = (\rho^2 + a^2)\Omega/2a^2$. Because the multiplicative function is different for the temporal and spatial coordinates, it is obvious that such a space does not preserve the Minkowski angles between *world lines*. However, all spatial coordinates are multiplied by the same Ω^2, and therefore angles between *physical lengths* are the same as those measured in flat space.

We may now assign an equivalent refractive index to the curved space described by (45). We proceed by varying the Action for the null geodesic [25]

$$\delta \int g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0. \tag{46}$$

This equation results in the following variation

$$\delta x^0 = \delta \int \dot{x}^0 dq = 0, \tag{47}$$

If the geometry is static, then $g_{0i} = 0$ and $\dot{x}^0$ can be easily determined from the line element itself

$$\dot{x}^0 = \left[-\frac{1}{g_{00}} \left(g_{ij}\dot{x}^i \dot{x}^j\right)\right]^{1/2}. \tag{48}$$

Plugging (48) into (47), and replacing the affine parameter q, with the spatial curve parameter ℓ, yields

$$\delta x^0 = \delta \int \left(-\frac{g_{ij}}{g_{00}} \frac{dx^i}{d\ell}\frac{dx^j}{d\ell}\right)^{1/2} d\ell = 0. \tag{49}$$

The metric given by (45) has a spatial component that is conformally flat:

$$g_{ij}dx^i dx^j = \Omega^2 d\ell'^2, \tag{50}$$

where $d\ell'^2 = d\rho^2 + \rho^2 d\phi^2$ denotes the flat space line element. Using this relation in (49) yields

$$\delta x^0 = \delta \int \frac{\Omega}{\Sigma} d\ell' = \delta \int n' d\ell' = 0. \tag{51}$$

This is simply Fermat's principle with effective refractive index $n' = \Omega/\Sigma$. For the metric given by (45), we find that

$$n' = \frac{2a^2}{\rho^2 + a^2}. \tag{52}$$

Thus, the curved spacetime induced by the metamaterial behaves like an isotropic, inhomogeneous refractive index profile.

V. COVARIANCE OF MAXWELL'S EQUATIONS

We shall now show an alternate method for calculating the permittivity tensor by exploiting the covariant nature of Maxwell's equations as outlined in [6, 9]. The analysis starts by expressing Maxwell's equations in generally covariant form (using SI units)

$$F_{[\mu\nu\,;\,\lambda]} = 0, \qquad \epsilon_0 F^{\mu\nu}{}_{;\,\nu} = j^\mu, \tag{53}$$

where $F^{\mu\nu}$ is the skew-symmetric field-strength tensor, and j^μ is the 4-current vector. In the expressions above, brackets denote alternation of the indices, while semicolons denote covariant differentiation. By virtue of the antisymmetry of $F^{\mu\nu}$, (53) can be expressed in terms of partial derivatives

$$F_{[\mu\nu\,,\,\lambda]} = 0, \qquad \epsilon_0 \left(\sqrt{-g} F^{\mu\nu}\right)_{,\,\nu} = \sqrt{-g} j^\nu, \tag{54}$$

where $g = det(g_{\alpha\beta})$. The dielectric analog to the field-strength tensor $F^{\mu\nu}$ is defined as

$$H^{\mu\nu} = \epsilon_0 \sqrt{-g} g^{\mu\lambda} g^{\nu\rho} F_{\lambda\rho}, \tag{55}$$

and is comprised of the displacement and auxiliary fields, $\boldsymbol{D}$ and $\boldsymbol{H}$. Expressing $F_{\mu\nu}$ in terms of $H^{\mu\nu}$ allows us to write (54) as the *macroscopic* Maxwell's equations in Cartesian coordinates

$$F_{[\mu\nu\,,\,\lambda]} = 0, \qquad H^{\mu\nu}{}_{;\,\nu} = J^{\mu}, \tag{56}$$

provided that the Plebanski constituent relations are satisfied

$$D_i = -\epsilon_0 \frac{\sqrt{-g}}{g_{00}} g^{ij} E^j + \frac{g_{j0}}{c g_{00}} \epsilon_{ijk} H^k, \qquad B_i = -\frac{\sqrt{-g}}{\epsilon_0 c^2 g_{00}} g^{ij} H^j - \frac{g_{j0}}{c g_{00}} \epsilon_{ijk} E_k. \tag{57}$$

Note that in (56), the geometrically scaled four-current vector is given by $J^{\mu} = \sqrt{-g} j^{\mu}$. In vector form, (57) reads

$$\boldsymbol{D} = \epsilon_0 \epsilon \boldsymbol{E} + \frac{\boldsymbol{w}}{c} \times \boldsymbol{H}, \qquad \boldsymbol{B} = \mu_0 \mu \boldsymbol{H} - \frac{\boldsymbol{w}}{c} \times \boldsymbol{E}. \tag{58}$$

The material parameters are related to the geometry via the following equations

$$\epsilon^{ij} = \mu^{ij} = -\frac{\sqrt{-g}}{\sqrt{\gamma} g_{00}} g^{ij}, \qquad w_i = \frac{g_{0i}}{\sqrt{\gamma} g_{00}}, \tag{59}$$

where $\gamma = det(\gamma_{ij})$ is the determinant of the metric describing the original ambient curvilinear coordinate system, and appears in the equation above in order to correct for the fact that the initial coordinate system may be something other than Cartesian [6]. Thus, Maxwell's equations in curved space can be expressed as the macroscopic Maxwell's equations in *flat* space inside a bi-anisotropic medium, whose material parameters are described by (59). Alternatively, (59) may be interpreted as the material parameters necessary to give rise to a curved geometry described by $g_{\alpha\beta}$. Incidentally, the bi-anisotropy vector $\boldsymbol{w}$ is responsible for coupling the electric and magnetic fields and is closely related to the velocity of a moving dielectric, though for our diagonal metric, $\boldsymbol{w} = 0$. Note that (59) is only valid in four dimensional spacetime, and we must make a modification for our use in three dimensions, as we restrict ourselves to one particular plane. When the spatial metric is conformally flat, as in (45), equation (59) must be augmented by a factor of Ω:

$$\epsilon^{ij} = \mu^{ij} = -\frac{\sqrt{-g\Omega^2}}{\sqrt{\gamma} g_{00}} g^{ij}. \tag{60}$$

The relevant parameters for the system described in Sec. III and IV are [26]

$$
\begin{aligned}
g^{ij} &= \frac{1}{\Omega^2} diag \left[1 \ , \ \frac{1}{\rho^2} \right] , \\
g &= -\Sigma^2 \Omega^4 \rho^2 , \\
\gamma^{ij} &= diag \left[1 \ , \ \frac{1}{\rho^2} \right] , \\
\gamma &= \rho^2 .
\end{aligned}
\tag{61}
$$

Finally, we may use these values in (60) to calculate the permittivity and permeability tensors

$$
\epsilon^{ij} = \mu^{ij} = \frac{\sqrt{\Sigma^2 \Omega^4 \Omega^2 \rho^2}}{\sqrt{\rho^2 \Sigma^2}} \frac{1}{\Omega^2} diag \left[1 \ , \ \frac{1}{\rho^2} \right] = \frac{\Omega}{\Sigma} \gamma^{ij} . \tag{62}
$$

These are the parameters that will give rise to Keplerian orbits for light rays inside the metamaterial, and are in accordance with (52).

VI. CONCLUSION

We have shown an alternative method for calculating the material properties of transformation media, by determining the metric that derives the desired equations of motion for a light ray inside the material. This method was demonstrated to yield the correct result in the case of the cylindrical cloaking device, and it generally provides a powerful way of approaching problems for which the transformations are not inherently obvious, in which case the process of choosing a transformation is circumvented.

In the event that the equations of motion lead to a curved spacetime, as in the case of the Keplerian orbit device, the material parameters cannot be calculated via the standard techniques of transformation optics, though one can easily apply the Plebanski constituent relations to determine these parameters. Moreover, Fermat's principle becomes particularly simple when the space is conformally flat. Incidentally, it should be noted that conformal flatness (and constant curvature) can be imposed directly by demanding that the Weyl conformal curvature tensor vanish in the equation of motion [24]. Lastly, further work should be carried out to explore the idea of embedding the curved spaces described by metamaterial in higher dimensional Euclidean spaces to calculate the material parameters.

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