1. a) Differentiate both sides of the given equation with respect to \( a \):

\[
\frac{d}{da} \left( \int_0^{\infty} e^{-ax^2} \, dx \right) = \frac{d}{da} \left( \frac{\pi}{a} \right)^{\frac{1}{2}}
\]

\[
= \int_0^{\infty} \frac{d}{da} \left( e^{-ax^2} \right) \, dx = -\frac{1}{2} \pi^{\frac{1}{2}} \left( \frac{1}{a} \right)^{\frac{3}{2}}
\]

\[
= -\int_0^{\infty} x^2 e^{-ax^2} \, dx = -\frac{1}{2} \pi^{\frac{1}{2}} \left( \frac{1}{a} \right)^{\frac{3}{2}}
\]

\[
= \int_0^{\infty} x^2 e^{-ax^2} \, dx = \frac{1}{2} \pi^{\frac{1}{2}} \left( \frac{1}{a} \right)^{\frac{3}{2}}
\]

(1)

And if we continue this process, we see that a general formula is:

\[
\int_0^{\infty} x^n e^{-ax^2} \, dx = \frac{(n-1)!!}{2^n} \left( \frac{\pi}{a} \right)^{\frac{1}{2}} \left( \frac{1}{a} \right)^{\frac{n+1}{2}}
\]
b) If we factor out an a, and then complete the square in the exponential, we see:

\[ \int_0^\infty e^{-ax^2} e^{-bx} \, dx = \int_0^\infty e^{-a(x^2 + \frac{b}{a})} \, dx \]

\[ = \int_0^\infty e^{-a(x^2 + \frac{b}{a} + \frac{b^2}{2a})} \, dx \]

\[ = e^{-\frac{b^2}{2a}} \int_0^\infty e^{-a(x^2 + \frac{b}{a} + \frac{b^2}{4a})} \, dx \]

\[ = e^{-\frac{b^2}{2a}} \int_0^\infty e^{-a(x^2 + \frac{b}{a} + \frac{b^2}{4a})} \, dx \]

\[ = e^{-\frac{b^2}{2a}} \int_0^\infty e^{-a(x^2 + \frac{b}{2a})} \, dx \]

(3)

Now, we make a small change of variables, \( w = (x + \frac{b}{2a}) \) so that \( dx = dw \) and the above integral is now:

\[ e^{-\frac{b^2}{2a}} \int_0^\infty e^{-a(x^2 + \frac{b^2}{4a})} \, dx = e^{-\frac{b^2}{2a}} \int_0^\infty e^{-aw^2} \, dw \]

\[ = \left( \frac{\pi}{a} \right)^{\frac{1}{2}} e^{-\frac{b^2}{4a}} \]

(4)

2. Using the formula for \( < E^2 > \) we can rewrite \( < E^2 > \) as:

\[ \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} = \frac{1}{Z} \frac{\partial}{\partial \beta} \left( \frac{\partial Z}{\partial \beta} \right) \]

(5)

Now using the reverse chain rule since \( Z \) is a function of \( \beta \) then the above is:

\[ \frac{1}{Z} \frac{\partial}{\partial \beta} \left( \frac{\partial Z}{\partial \beta} \right) = \frac{\partial}{\partial \beta} \left( -Z \frac{\partial \ln(Z)}{\partial \beta} \right) \]

\[ = \frac{1}{Z} \left[ -Z \frac{\partial^2 \ln(Z)}{\partial \beta^2} - \frac{\partial Z}{\partial \beta} \frac{\partial \ln(Z)}{\partial \beta} \right] \]

\[ = \frac{\partial^2 \ln(Z)}{\partial \beta^2} - \frac{\partial Z}{\partial \beta} \frac{\partial \ln(Z)}{\partial \beta} \]

\[ = \frac{\partial^2 \ln(Z)}{\partial \beta^2} + \left( \frac{\partial \ln(Z)}{\partial \beta} \right)^2 \]

\[ = \frac{\partial^2 \ln(Z)}{\partial \beta^2} + < E > \]

\[ = \frac{\partial < E >}{\partial \beta} + < E > \]

(6)

by using the formula for the average energy given. Thus, we see that the mean square fluctuations are:

\[ \sigma_E = < E^2 > - < E >^2 = \frac{\partial < E >}{\partial \beta} \]
thus, we now see that $\beta = \frac{1}{kT}$ so $T = \frac{1}{k\beta}$. Thus, the chain rule gives:

$$\sigma_E = \langle E^0 \rangle - \langle E \rangle^2 = \frac{\partial \langle E \rangle}{\partial \beta} \cdot \frac{1}{k\beta^2} \cdot \langle E \rangle = kT \frac{\partial \langle E \rangle}{\partial T}$$

which is in terms of k, T and $c_v$ as desired.

3. 

$$dF = dU - TdS - SdT = TdS - PdV + \mu dN - TdS - SdT = -SdT - PdV + \mu dN$$

Now,

$$dF = \frac{\partial F}{\partial T}dT + \frac{\partial F}{\partial V}dV + \frac{\partial F}{\partial N}dN$$

but these must be equal, so we form the Thermodynamic relations:

$$\frac{\partial F}{\partial T} = -S, \quad \frac{\partial F}{\partial V} = -P, \quad \frac{\partial F}{\partial N} = \mu$$

4. a) From problem three, we can use the partial relations and the fact that partials commute to see:

$$\frac{\partial P}{\partial T} = \frac{\partial}{\partial T} (P) = \frac{\partial}{\partial T} \left( \frac{\partial F}{\partial V} \right) = \frac{\partial}{\partial V} \left( \frac{\partial F}{\partial T} \right) = \frac{\partial}{\partial V} (S)$$

as desired.

b) First we must calculate F, using the formula given,

$$F(T, V, N) = -kT \ln \left( \frac{4V}{k^3} \left( 2\pi mkT \right)^{\frac{3}{2}} \right)^N$$

$$= -NkT \left[ \ln(4) + \ln(V) + \frac{3}{2} \ln(2\pi m) + \frac{3}{2} \ln(kT) - 3\ln(h) \right]$$

(10)
Thus,
\[ P = \frac{\partial F}{\partial V} = \frac{\partial}{\partial V} \left( -NkT \left[ \ln(4) + \ln(V) + \frac{3}{2} \ln(2\pi m) + \frac{3}{2} \ln(kT) - 3\ln(h) \right] \right) \]
\[ = \frac{\partial}{\partial V} (-NkT[\ln(V)]) \]
\[ = -NkT \frac{\partial \ln(V)}{\partial V} \]
\[ = \frac{NkT}{V} \]
(11)
Which if we just multiply both sides by V gives us the familiar ideal gas law
\[ PV = NkT \]

c) Using the above, and the relations from problem 2, we see that:
\[ < E > = \frac{\partial \ln Z}{\partial \beta} \]
\[ = \frac{\partial}{\partial \beta} \left( N \left[ \ln(4) + \ln(V) + \frac{3}{2} \ln(2\pi m) + \frac{3}{2} \ln(kT) - 3\ln(h) \right] \right) \]
\[ = \frac{\partial}{\partial \beta} \left( N[\ln(4) + \ln(V) + \frac{3}{2} \ln(2\pi m) - \frac{3}{2} \ln(\beta) - 3\ln(h)] \right) \]
\[ = \frac{3}{2\beta} \]
\[ = \frac{3}{2} NKT \]
(12)
Which matches the equipartition theorem as desired. Finally:
\[ c_v = \frac{\partial < E >}{\partial T} \]
\[ = \frac{\partial}{\partial T} \left( \frac{3}{2} NKT \right) \]
\[ = \frac{3}{2} NK \]
(13)