PHYSICS 298 - HW 1 SOLUTIONS

1. a)Differentiate both sides of the given equation with respect to a:

$$\frac{d}{da}\left(\int_0^\infty e^{-ax^2}dx\right) = \frac{d}{da}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}$$
$$= \int_0^\infty \frac{d}{da}\left(e^{-ax^2}dx\right) = -\frac{1}{2}\pi^{\frac{1}{2}}\left(\frac{1}{a}\right)^{\frac{3}{2}}$$
$$= -\int_0^\infty x^2e^{-ax^2}dx = -\frac{1}{2}\pi^{\frac{1}{2}}\left(\frac{1}{a}\right)^{\frac{3}{2}}$$
$$= \int_0^\infty x^2e^{-ax^2}dx = \frac{1}{2}\pi^{\frac{1}{2}}\left(\frac{1}{a}\right)^{\frac{3}{2}}$$

(1)

So, diff. again to see:

$$\frac{d^2}{da^2}\left(\int_0^\infty e^{-ax^2}dx\right) = \frac{d^2}{da^2}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}$$
$$= \int_0^\infty \frac{d^2}{da^2}\left(e^{-ax^2}dx\right) = -\frac{3}{4}\pi^{\frac{1}{2}}\left(\frac{1}{a}\right)^{\frac{5}{2}}$$
$$= -\int_0^\infty x^4e^{-ax^2}dx = -\frac{3}{4}\pi^{\frac{1}{2}}\left(\frac{1}{a}\right)^{\frac{3}{2}}$$
$$= \int_0^\infty x^4e^{-ax^2}dx = \frac{3}{4}\pi^{\frac{1}{2}}\left(\frac{1}{a}\right)^{\frac{3}{4}}$$

(2)

And if we continue this process, we see that a general formula is:

$$\int_0^\infty x^ne^{-ax^2}dx = \frac{(n-1)!!}{2^{\frac{n}{2}}}(\pi)^{\frac{1}{2}}\left(\frac{1}{a}\right)^{\frac{n-1}{2}}$$

b)If we factor out an a, and then complete the square in the exponential, we see:

$$
\begin{aligned}
&= \int_0^\infty e^{-ax^2} e^{-bx} dx = \int_0^\infty e^{-a(x^2+\frac{b}{a})} dx \\
&= \int_0^\infty e^{-a(x^2+\frac{b}{a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2})} dx \\
&= \int_0^\infty e^{-a(x^2+\frac{b}{a}+\frac{b^2}{4a^2}} e^{-\frac{b^2}{4a^2})} dx \\
&= e^{-\frac{b^2}{4a}} \int_0^\infty e^{-a(x^2+\frac{b^2}{a}+\frac{b^2}{4a^2})} dx \\
&= e^{-\frac{b^2}{4a}} \int_0^\infty e^{-a(x+\frac{b^2}{2a})^2} dx
\end{aligned}
\tag{3}
$$

Now, we make a small change of variables, $w = (x + \frac{b^2}{2a})$ so that $dx = dw$ and the above integral is now:

$$
\begin{aligned}
e^{-\frac{b^2}{4a}} \int_0^\infty e^{-a(x+\frac{b^2}{2a})^2} dx =& e^{-\frac{b^2}{4a}} \int_0^\infty e^{-aw^2} dw \\
=& \left(\frac{\pi}{a}\right)^{\frac{1}{2}} e^{-\frac{b^2}{4a}}
\end{aligned}
\tag{4}
$$

2. Using the formula for $< E^2 >$ we can rewrite $< E^2 >$ as:

$$
\frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2} = \frac{1}{Z}\frac{\partial}{\partial \beta}\left(\frac{\partial Z}{\partial \beta}\right)
\tag{5}
$$

Now using the reverse chain rule since Z is a function of β then the above is:

$$
\begin{aligned}
\frac{1}{Z}\frac{\partial}{\partial \beta}\left(\frac{\partial Z}{\partial \beta}\right) &= \frac{1}{Z}\frac{\partial}{\partial \beta}\left(-Z\frac{\partial ln(Z)}{\partial \beta}\right) \\
&= \frac{1}{Z}\left[-Z\frac{\partial^2 ln(Z)}{\partial \beta^2} - \frac{\partial Z}{\partial \beta}\frac{\partial ln(Z)}{\partial \beta}\right] \\
&= -\frac{\partial^2 ln(Z)}{\partial \beta^2} - \frac{1}{Z}\frac{\partial Z}{\partial \beta}\frac{\partial ln(Z)}{\partial \beta} \\
&= -\frac{\partial^2 ln(Z)}{\partial \beta^2} + \left(\frac{\partial ln(Z)}{\partial \beta}\right)^2 \\
&= -\frac{\partial^2 ln(Z)}{\partial \beta^2} + < E >^2 \\
&= \frac{\partial < E >}{\partial \beta} + < E >^2
\end{aligned}
\tag{6}
$$

by using the formula for the average energy given. Thus, we see that the mean square fluctuations are:

$$
\sigma_E = < E^@ > - < E >^2 = \frac{\partial < E >}{\partial \beta}
$$

thus, we now see that $\beta = \frac{1}{kT}$ so $T = \frac{1}{k\beta}$. Thus, the chain rule gives:

$$\begin{aligned}\sigma_E = < E^{@} > - < E >^2 = \frac{\partial < E >}{\partial \beta} &= \frac{\partial T}{\partial \beta}\frac{\partial < E >}{\partial T} \\ &= \frac{1}{k\beta^2}\frac{\partial < E >}{\partial T} \\ &= kT^2\frac{\partial < E >}{\partial T} \\ &= kT^2 c_v\end{aligned} \tag{7}$$

which is in terms of k, T and c_v as desired.

3.

$$\begin{aligned} dF &= dU - TdS - SdT \\ &= TdS - PdV + \mu dN - TdS - SdT \\ &= -Sdt - PdV + \mu dN \end{aligned} \tag{8}$$

Now,

$$dF = \frac{\partial F}{\partial T}dT + \frac{\partial F}{\partial V}dV + \frac{\partial F}{\partial N}dN$$

but these must be equal, so we form the Thermodynamic relations:

$$\frac{\partial F}{\partial T} = -S \quad , \quad \frac{\partial F}{\partial V} = -P \quad , \quad \frac{\partial F}{\partial N} = \mu$$

4. a) From problem three, we can use the partial relations and the fact that partials commute to see:

$$\begin{aligned}\frac{\partial P}{\partial T} &= \frac{\partial}{\partial T}(P) \\ &= \frac{\partial}{\partial T}\left(\frac{\partial F}{\partial V}\right) \\ &= \frac{\partial}{\partial V}\left(\frac{\partial F}{\partial T}\right) \\ &= \frac{\partial}{\partial V}(S)\end{aligned} \tag{9}$$

as desired.

b) First we must calculate F, using the formula given,

$$\begin{aligned} F(T,V,N) &= -kTln\left[\left(\frac{4V}{h^3}(2\pi mkT)^{\frac{3}{2}}\right)^N\right] \\ &= -NkT\left[ln(4) + ln(V) + \frac{3}{2}ln(2\pi m) + \frac{3}{2}ln(kT) - 3ln(h)\right]\end{aligned} \tag{10}$$

Thus,

$$P = \frac{\partial F}{\partial V} = \frac{\partial}{\partial V}\left(-NkT\left[ln(4) + ln(V) + \frac{3}{2}ln(2\pi m) + \frac{3}{2}ln(kT) - 3ln(h)\right]\right)$$
$$= \frac{\partial}{\partial V}\left(-NkT[ln(V)]\right)$$
$$= -NkT\frac{\partial ln(V)}{\partial V}$$
$$= \frac{NkT}{V}$$

(11)

Which if we just multiply both sides by V gives us tha familiar ideal gas law $PV = NkT$

c) Using the above, and the relations from problem 2, we see that:

$$< E > = \frac{\partial lnZ}{\partial \beta}$$
$$= \frac{\partial}{\partial \beta}\left(N\left[ln(4) + ln(V) + \frac{3}{2}ln(2\pi m) + \frac{3}{2}ln(kT) - 3ln(h)\right]\right)$$
$$= \frac{\partial}{\partial \beta}\left(N[ln(4) + ln(V) + \frac{3}{2}ln(2\pi m) - \frac{3}{2}ln(\beta) - 3ln(h)\right]\Bigg)$$
$$= \frac{3}{2\beta}N$$

(12) $$= \frac{3}{2}NKT$$

Which matches the equipartition theorem as desired. Finally:

$$c_v = \frac{\partial < E >}{\partial T}$$
$$= \frac{\partial}{\partial T}\left(\frac{3}{2}NKT\right)$$
$$= \frac{3}{2}NK$$

(13)