Preliminary Examination: Statistical Mechanics, 08/21/2018

Answer a total **THREE** questions out of **FOUR**. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book, or on consecutively numbered sheets of paper stapled together. Make sure you clearly indicate who you are and the problem you are solving. Double-check that you include everything you want graded, and nothing else.

1. A single harmonic oscillator has the energy $\epsilon = (n + \frac{1}{2})h\nu$. Assume N distinguishable oscillators have the total energy $E_M = \frac{1}{2}Nh\nu + Mh\nu$, that is, there are M excitations shared between the N oscillators.

   a) What is the number of microstates for the macrostate with M excitations? What, then, is the entropy of the system with energy E_M? (You can assume that both $N, M \gg 1$.)

   b) Use the result from part (a) to show that the relationship between energy and temperature is
   $$E = Nh\nu\left(\frac{1}{2} + \frac{1}{e^{\frac{h\nu}{kT}} - 1}\right).$$

   Hint: You might need Stirling's formula for $n \gg 1$,
   $$\ln(n!) \approx n\ln n - n.$$

2. Show that there is no Bose-Einstein condensation in an ideal two-dimensional gas of massive bosons. In order to do so,

   a) show that the total number of particles in a two-dimensional region $L_x \times L_y$ can be written as
   $$N = 2\pi L_x L_y \frac{m}{h^2}\int_0^\infty \frac{d\epsilon}{e^{\frac{\epsilon-\mu}{kT}} - 1}.$$

   b) The integrand of the above equation can now be rewritten as an infinite sum, and for each term in the sum the integral can be resolved. Use the resulting infinite sum to show that there cannot be a chemical potential μ for which a macroscopic population (of order N) occurs.

   Hint: you might find the following Taylor expansion useful for small x:
   $$\frac{1}{1-x} = \sum_{p=0}^{\infty} x^p.$$

3. An ideal two-Dimensional (2D) classical gas of N particles is contained inside the impenetrable circle of radius R at constant temperature T. The 2D central potential well $U(r)$ is located at the center of this circle,

$$U(r) = \begin{cases} -U_0 & : \quad r < r_0 \\ 0 & : \quad r \geq r_0 \end{cases}$$

where r_0 ($r_0 \ll R$) and U_0 ($U_0 > 0$) are the radius and depth of the potential well, respectively. Classical particles are distinguishable, but they have identical masses m.

a) Determine the canonical partition function $Z(N, T)$.

b) What is the meaning of pressure in two dimensions? Find the pressure $p(N, T)$ for this system and use that to determine the Gibbs free energy $F(N, T)$.

c) Calculate the average particle energy $< \varepsilon >$.

d) Find the average number of particles inside the potential well and particles' average potential energy $< U >$.

4. A completely degenerate Fermi gas of N neutral atoms with atomic mass m and spin $s = 3/2$ occupies a large spherical volume V at zero temperature. Calculate the Fermi momentum p_F, the gas total energy E and the average frequency γ of particle collisions with the spherical surface wall, if :

a) the dispersion relation between the fermion's $\varepsilon(\vec{p})$ and momentum $\vec{p}$ is given by the classical formula $\varepsilon(\vec{p}) = p^2/2m$, where m is the particle mass;

b) the fermions are ultra-relativistic particles with the dispersion law $\varepsilon(\vec{p}) = |\vec{p}|c$, where c is the speed of light.

Hint: The average frequency (rate) of collisions between particles and the surface σ is defined as $\gamma = \frac{1}{4} < v > n\sigma$, where $v = \partial\varepsilon(p)/\partial p$ is the particle velocity and n is the particle number density. The average velocity is calculated using the momentum distribution function. This expression is valid both for the classical and degenerate quantum gases.