Preliminary Exam: Statistical Mechanics, Tuesday January 13, 2015, 9:00-12:00

Answer a total of any **THREE** out of the four questions.

For your answers you can use either the blue books or individual sheets of paper.

If you use the blue books, put the solution to each problem in a separate book.

If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set.

Be sure to put your name on each book and on each sheet of paper that you submit.

If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

Problem 1

You discover a new state of matter that has equation of state and internal energy given by

$$p \;=\; A\,\frac{T^3}{V}, \qquad U \;=\; B\,T^n \ln\frac{V}{V_0} + f(T),$$

where A, B, n, V_0 are constants while $f(T)$ only depends on the temperature. Find a numerical value for n, and find a relation between the constants B and A.
Hint: Find a Legendre transformation of the entropy $S(U, V)$ to a function that only depends on T and V

Problem 2

Consider a two-dimensional gas of noninteracting fermions with spin s with energy dispersion given by the usual relativistic expression for a particle of mass m

$$\epsilon(\mathbf{k}) \;=\; \sqrt{(mc^2)^2 + (\hbar c|\mathbf{k}|)^2}.$$

The gas consists of N particles in an area A and so has number density $n = N/A$.

a) Find an expression for the Fermi wave vector k_F for this gas as a function of number density n.

b) Show the expression for the total energy of the gas at $T = 0$. Show that this contains the (purely algebraic) expression

$$\int_0^{x_F} x\sqrt{x^2+1}\,dx.$$

What is x_F?

c) Consider the nonrelativistic limit $\hbar k_F \ll mc$. Find the lowest order two terms in x to obtain an expression for the total energy valid in this limit. Express your answer in terms of N and A and show that it has the form $E = N(mc^2 + \epsilon_{\mathrm{NR}})$ where ϵ_{NR} is the energy density of a nonrelativistic gas of fermions.

d) Now consider the ultrarelativistic limit $\hbar k_F \gg mc$. Obtain an expression for the total energy valid in this limit.

Problem 3

Consider the transverse modes of a quantum mechanical string of length L and uniform mass per unit length μ stretched with tension F between two fixed points. The string is in thermal equilibrium at temperature T. The transverse size of the string is negligible compared to its length, and the effects of gravity on its motion are negligible. Consider transverse displacement $\vec{y}$ of the string, small compared to L, such that it obeys the linear equation of motion

$$\mu\frac{d^2\vec{y}}{dt^2} = F\frac{d^2\vec{y}}{d^2x} \tag{1}$$

The solutions to this wave equation are of the form

$$\vec{y}(x,t) = (a_k\hat{e}_1 + b_k\hat{e}_2)\sin(kx - \omega t) \tag{2}$$

where the sum is over all wavenumber k values that obey the boundary conditions of the string, and $\omega = vk$ where $v = \sqrt{F/\mu}$ is the transverse wave velocity on the string.

a) What is the energy of a single quantum of excitation in a mode with wavenumber k?

b) Write down the grand canonical partition function for the system.

c) What is the heat capacity C of the system in the limit $k_{\scriptscriptstyle B}T \gg \hbar\omega_1$ is the frequency of the fundamental mode?

Hint: You may find the following integral to be useful.

$$\int_0^\infty \frac{x}{e^x - 1}dx = \frac{\pi^2}{6} \tag{3}$$

Problem 4

A refrigerator consists of a system of a large number N of distinguishable atoms placed in a variable uniform external magnetic field of magnitude B. The atoms have a total electronic spin of J and maximum magnetic moment $\mu_{\text{max}} = \mu_0 J$. The closed thermal cycle works as follows.

1. The magnetic field is ramped up adiabatically until the temperature reaches T_2 at field B_2.
2. The system is brought into thermal contact with the hot reservoir at temperature T_2, and the magnetic field is ramped up further to maximum field B_{max} expelling Q_2 as heat.
3. The system is removed from thermal contact with the hot reservoir and the magnetic field is ramped adiabatically down until the temperature reaches T_1 at field B_1.
4. The system is brought into thermal contact with the cold side at temperature T_1, and the magnetic field is ramped down to B_{min}, absorbing Q_4 as heat.
5. The system is removed from thermal contact with the cold reservoir and step 1 resumes.

The atoms interact with the external field but not with one another. The interaction of one atom with the external field is given by $H_{\text{int}} = -\mu_0 m B$ where m labels the projection of the atom's spin onto the magnetic field axis.

a) What is the entropy of the system in the limit $B \to 0$ at fixed temperature? Hint: Use the expression $S = -k_B \sum_i P_i \log P_i$ (P_i is the probability for configuration i), and make a general argument based on degeneracy of states. If you attempt to compute a closed form expression for S at finite B and take the limit of large field, you will find the calculation to be cumbersome.

b) What is the entropy of the system in the limit of strong field B at fixed temperature? What is the field strength B_0 that distinguishes the strong and weak field limits?

c) Suppose field $B_1 \ll B_0$ and $B_2 \gg B_0$. During steps 1 and 3 the magnetic field B is changing when the system is not in thermal contact with either reservoir, so $\Delta S = 0$ during those steps. Does this contradict the results from parts (a) and (b)? If not, what condition must be satisfied?

d) Derive the maximum theoretical efficiency that can be obtained using this refrigerator cycle, defined as $\varepsilon = (Q_2 - Q_4)/Q_2$.