Statistical Mechanics / Quantum Mechanics

General Exam Questions for August 26, 2005

Instructions

Answer two questions from the Statistical Mechanics section and four questions from the Quantum Mechanics section, for a total of six problems. Put each of your solutions in a separate answer book. Make sure that you label and sign your name on the cover of each book.

Some Useful Relations

$$\ln N! \to N \ln N - N \qquad \text{as N} \to \infty$$

Coherent state

$$|\alpha\rangle = e^{-\alpha(a^\dagger - a)}|0\rangle, \qquad e^{\alpha(a^\dagger - a)} a e^{-\alpha(a^\dagger - a)} = a + \alpha$$

spherical Bessels:

$$j_l(r) = R_l(r)\frac{\sin r}{r} + S_l(r)\frac{\cos r}{r}, \qquad n_l(r) = R_l(r)\frac{\cos r}{r} - S_l(r)\frac{\sin r}{r},$$

$$\text{where} \quad R_l(r) + iS_l(r) = \sum_{s=0}^{l} \frac{i^{s-l}(l+s)!}{2^s s!(l-s)!} r^{-s},$$

$$\text{and with asymptotic behavior} \quad j_\ell(r) \to \frac{\sin(r - \ell\pi/2)}{r}, \quad n_\ell(r) \to -\frac{\cos(r - \ell\pi/2)}{r}.$$

Plane wave:

$$e^{ikz} = \Sigma_{l=0}^{\infty}(2l+1)i^l j_l(kr) P_l(\cos\theta)$$

I. Statistical Mechanics

1. Consider four identical particles which must be distributed over four motional energy levels.
   a) How many distinct quantum states exist for this system if the particles have spin $I = 5/2$?
   b) How many distinct quantum states exist for this system if the particles have spin $I = 2$?

2. Consider a one-dimensional gas of N particles that are confined to move on a closed loop of length L. The particles interact pairwise with a potential

$$u_{ij}(r) = \begin{cases} \infty, & r < \sigma \\ -\frac{\varepsilon}{r^6}, & \sigma \leq r < L/2, \end{cases}$$

   where r is the distance between two particles measured along the loop. The number of particles N in the system is large but finite, such that $1 \ll N < L/2\sigma$.

   a) Evaluate the total potential energy U of this system. You may assume that the particles move in an uncorrelated fashion, i.e. each particle interacts with all others as if they were evenly distributed over the range $\sigma < |r| < L/2$ (the "mean field" approximation).
   b) Derive the canonical partition function Z for this system, assuming a density $N/L < 1/2\sigma$.

   (b) What is the chemical potential of the particles as a function of particle density? Note that the chemical potential is defined as

$$\mu = \left(\frac{\partial A}{\partial N}\right)_{T,V}$$

   in terms of the Helmholtz free energy $A = -kT\,lnZ$.

   (c) Derive the equation of state within the mean field approximation. Note that in one dimension, the pressure is defined as

$$P = -\left(\frac{\partial A}{\partial L}\right)_{N,T},$$

   where A is the free energy. Do you expect this system to exhibit a phase transition at some densities and temperatures? Why?

3. Consider a system of identical bosonic particles. Each boson is a bound state of two identical spin-1/2 fermions with total spin 0. The bosons sit on a regular lattice of sites with a uniform density of $1/v_b$ with total volume V. Once they are ionized, the free fermions sit on a different lattice of sites with a larger density $1/v_f$ but the same total volume V. The system is in equilibrium temperature T. At low temperatures the binding is very strong so that there are no free fermions in the ensemble. As the temperature is raised, at $T = T_c$ the boson gas undergoes a phase transition due to ionization and the constituent fermions become free. This transition is of the first order. You may assume that the fermions and bosons are both heavy, so that at the relevant temperatures their kinetic energy vanishes.

   a) What is the entropy S_b of the boson gas at temperature T_c? What is the entropy S_f of the fermion gas at T_c?
   b) Calculate the latent heat of the transition in terms of the particle density of the fermions ρ, the critical temperature T_c, and the volumes v_b and v_f.
   c) What is the binding energy B of the bosons?

4. Consider a system of carbon nanotubes that are suspended in water. A carbon nanotube can be regarded as a uniform hollow cylinder with a moment of inertia I about its symmetry axis. The axial Brownian rotation of the nanotubes can be described by the differential equation

$$\frac{\partial \omega}{\partial t} + \gamma\omega = \frac{T_s(t)}{I},$$

where ω is the angular velocity, γ the friction coefficient, and I the moment of inertia of the nanotubes. $T_s(t)$ is a stochastic torque generated by random collisions of water molecules, with $\langle T_s \rangle = 0$.

   a) Show that, for long observation times, the mean square angular displacement is given by the expression

$$\langle \phi^2 \rangle = \frac{2}{\gamma}\langle \omega^2 \rangle t.$$

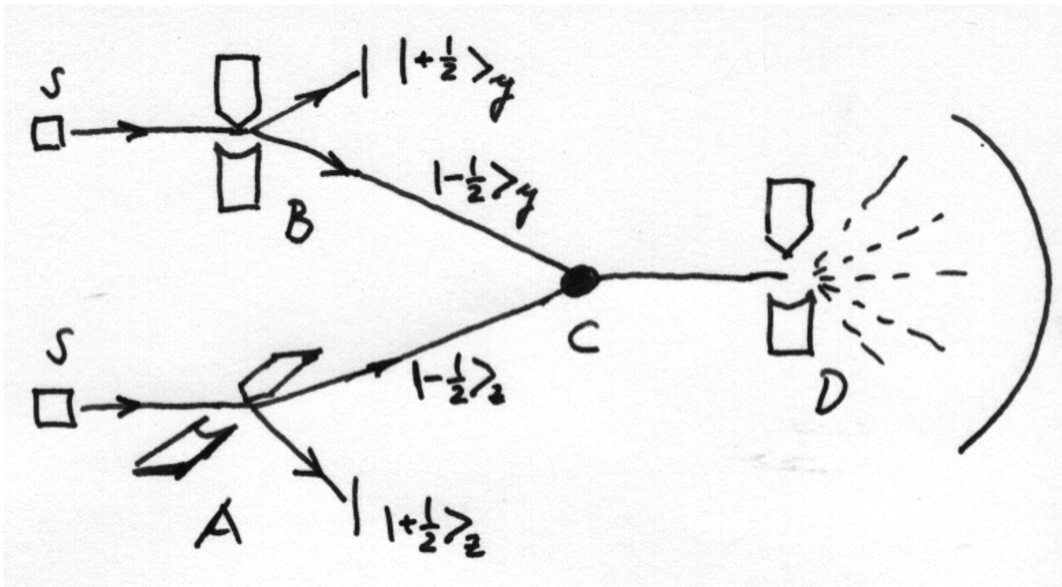
   (Hint: You may consider the tubes to be large enough that the average $\langle \phi T_s \rangle = 0$ but small enough that

$$\frac{d^2}{dt^2}\phi^2 \ll \gamma \frac{d}{dt}\phi^2$$

   b) Find the value of $\langle \omega^2 \rangle$ if the nanotubes are in thermal equilibrium with the water at temperature T.

II. Quantum Mechanics

1. An experimental setup consists of three Stern-Gerlach beam splitters, as shown in the illustration. Particles of spin $1/2$ emanate from the two sources, and are split into two beams in the z direction at point A, and in the y direction at point B. The $\left|-\frac{1}{2}\right\rangle_z$ and the $\left|-\frac{1}{2}\right\rangle_y$ beam are combined at point C. There the particles form a s-wave bound state through an interaction that preserves their individual spins. The beam of molecules is then passed through a third Stern-Gerlach splitter at point D, oriented in the y direction.



   a) Sketch the resulting pattern on the screen after point D. Label each trace with the appropriate quantum numbers.

   b) What are the relative intensities of the traces?

2. Consider a system containing N states of energy E_i ($i = 1, \ldots, N$, with $E_1 < E_2 < \cdots < E_N$) described by the time-independent Hamiltonian

$$H_0 = \sum_{i=1}^{N} E_i |i\rangle\langle i| \, .$$

A sinusoidal excitation (laser light with frequency ω) perturbs the system initially in the state $|1\rangle$. This interaction has the form

$$W(t) = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\gamma}{1 + (B - B_{ij})^2 / B_{12}^2} e^{i\omega t} |i\rangle\langle j| + \text{h.c.} \, ,$$

where γ and B_{ij}'s are real. Here B is an external magnetic field that can be tuned independently of the laser frequency. The system is in state 1 at $t = 0$.

a) Write down the equations that govern the evolution of the state for the coefficents $c_i(t) = e^{iE_i t/\hbar}\langle i|\psi(t)\rangle$

b) We want to couple only state $|1\rangle$ and $|2\rangle$ so we select $B = B_{12}$ to maximize W_{12} while suppressing the other states (assume $B_{12} \ll B_{ij}$ for all other i and j). Write down the resulting approximate two-level system equations to solve.

c) Solve for the probability of exciting $|2\rangle$, i.e. $|c_2(t)|^2$. Show that you recover the Rabi formula $|c_2(t)|^2 = \sin^2(\gamma t/\hbar)$ on resonance.

3. Consider an atom of hydrogen in a constant external electric field oriented along the z-axis.

a) Write down the perturbation W due to the field $\vec{F}$.

b) Find the first-order correction to the energy of the $1s$ level.

c) What is the polarizability α of the $1s$ state to first order in W? Give the expression, assuming that $E_1 - E_n \approx E_1 - E_2$. [Hint: (i) Use the closure relation; (ii)$\langle r^2\rangle = 3a_0^2$ where a_0 is the Bohr radius.]

4. Consider scattering of a quantum mechanical particle of mass m from a spherical square well potential

$$V(r) = \begin{cases} 0, & r > a \\ -V_0, & r < a \end{cases}$$

. A general solution of the Schroedinger equation for $r > a$ is

$$u(r,\theta) = \sum_{l=0}^{\infty} R_l(r) P_l(\cos\theta)$$

with

$$R_l(r) = A_l\left[\cos\delta_l\, j_l(kr) - \sin\delta_l\, n_l(kr)\right]$$

with j_l and n_l - spherical Bessel functions.

In a scattering problem we are looking for a solution which asymptotically (at large distances) has the form of an incoming plane wave plus a radially outgoing scattered wave

$$u(r,\theta) \rightarrow_{r\rightarrow\infty} C\left[e^{ikz} + \frac{1}{r} f(\theta) e^{ikr}\right]$$

The differential scattering cross section $\sigma(\theta)$ is related to f by

$$\sigma(\theta) = |f(\theta)|^2$$

a) Show that the two expressions for the wave function are compatible if the amplitudes of the partial waves are related to the phase shifts by

$$A_l = (2l+1)i^l e^{i\delta_l}$$

and the amplitude of the scattered wave is

$$f(\theta) = \frac{1}{2ik}\Sigma_{l=0}^{\infty}(2l+1)(e^{2i\delta_l} - 1)P_l(\cos\theta)$$

b) The phase shifts for our scattering problem are found by requiring that the ratio of slope to value of the wave function $\frac{1}{R_l(r)}\frac{dR_l}{dr}$ is continous across the boundary $x^2 = a^2$. This gives

$$\frac{k\left[j_l'(ka)\cos\delta_l - n_l'(ka)\sin\delta_l\right]}{j_l(ka)\cos\delta_l - n_l(ka)\sin\delta_l} = \gamma_l, \qquad \rightarrow \qquad \tan\delta_l = \frac{kj_l'(ka) - \gamma_l j_l(ka)}{kn_l'(ka) - \gamma_l n_l(ka)}$$

where γ_l is the ratio of slope to value of the interior wave function at $r = a$. Using the general form of the solution in the interior and the requirement that the wave function is finite at the origin, show that

$$\gamma_l = \frac{\alpha j_l'(\alpha a)}{j_l(\alpha a)}, \quad \alpha = \left[k^2 + \frac{2mV_0}{\hbar^2}\right]^{1/2}$$

c) Assume that at low energies ($ka \ll 1$) the cross section is dominated by the $l = 0$ partial wave,

$$\sigma(\theta) = \frac{1}{k^2}\sin^2\delta_0$$

Fixing k at some value with $\hbar k \ll (2mV_0)^{1/2}$, show that the cross section reaches a maximum value when the following relation is satisfied.

$$V_0 = \frac{\pi^2\hbar^2}{8ma^2}$$

5. A spherically symmetric quantum rotator is described by the Hamiltonian $H_0 = \frac{1}{2I}L_i^2$, where I is the moment of inertia and L_i is the component of angular momentum in the i'th direction. Consider such a rotator in a magnetic field in the third direction. The Hamiltonian of the system is

$$H = \frac{1}{2I}\mathbf{L}^2 + \mu L_3$$

a) Find the spectrum of the Hamiltonian (warning: the value of the total angular momentum is not fixed for this rotator).

b) Calculate the commutation relations of L_3 with $L_+ = L_1 + iL_2$ and $L_- = L_1 - iL_2$. Prove that L_+ acts as a raising operator (raises the eigenvalue of L_3 by one) and L_- acts as a lowering operator (lowers the eigenvalue of L_3 by one.

c) Show that in a high angular momentum representation $L^2 = l(l+1)$, $l \gg 1$ the operators $\frac{1}{\sqrt{2l\hbar}}L_\pm$ (to within accuracy $O(1/l)$) can be considered as harmonic oscillator raising and lowering operators a and $a^\dagger$ when they act on states close to the lowest m-state $l_3 = -l$. [Hint: Consider the commutation relations you have calculated in b.]

d) Suppose the magnetic field is large ($\mu I \gg \hbar$) and it also has a small horizontal component, so that the rotator Hamilton is

$$H' = \frac{1}{2I}\mathbf{L}^2 + \mu L_3 + \alpha\mu L_1$$

with $\alpha \ll 1$. What is the ground state of H'?

6. To find stationary states of a quantum mechanical Hamiltonian H we minimize the expectation value of H over the space of all normalized eigenfunctions $\psi(x)$. That is we need to minimize

$$\delta J = 0 \ \text{ with } \ J \equiv \int d^3r \psi^*(\vec{r}) H(\vec{r}, \vec{\nabla}) \psi(\vec{r}) \,, \tag{1}$$

under the constraint

$$\int d^3r \psi^*(\vec{r}) \psi(\vec{r}) = 1 \,. \tag{2}$$

Note: Eq.(1) is a statement that the energy of the system is stationary, and Eq.(2) is the constraint that there exists only one particle in the system.

a) For a general case of a functional $g(y_i(x), \frac{\partial y_i(x)}{\partial x_\alpha}, x_\alpha)$ of the degrees of freedom y_i which depend on the variables x_α the Euler-Lagrange equations are obtained by solving

$$\delta \int g\left(y_i, \frac{\partial y_i}{\partial x_\alpha}, x_\alpha\right) d^3x = 0 \,,$$

where g may include constraints, which leads to

$$\frac{\partial g}{\partial y_i} - \sum_j \frac{\partial}{\partial x_\alpha}\left(\frac{\partial g}{\partial(\partial y_i/\partial x_\alpha)}\right) = 0 \,.$$

Give an explicit form for g that enables the Euler-Lagrange equations to be used to find the extremum of J above, subject to the normalization constraint.

b) Consider that H is the Hamiltonian for a particle of mass m

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}) \, .$$

Assume that ψ and ψ^* vanish fast enough at ∞. Show that the Euler-Lagrange equations lead to the Schrödinger equation. What does the Lagrange multiplier that incorporates the normalization constraint in part (a) correspond to in this case?