Preliminary Exam: Quantum Mechanics, Friday January 16, 2015, 9:00-1:00

Answer a total of any **FOUR** out of the five questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than four problems, only the first four problems as listed on the exam will be graded.

Some possibly useful information:

$$\nabla^2 \;=\; \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \;=\; \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\text{Hermite polynomial} \;=\; H_n(x) \;=\; (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \;\;,\;\; H_0(x)=1 \;\;,\;\; H_1(x)=2x \;\;,\;\; H_2(x)=4x^2-2$$

$$\text{Laguerre} \;=\; L_n(r) \;=e^r\frac{d^n}{dr^n}\left(r^n e^{-r}\right) \;\;,\;\; \text{associated Laguerre} \;=\; L^q_{n+q}(r) \;=\; (-1)^q\frac{d^q}{dr^q}L_{n+q}(r) \;.$$

$$\text{Legendre polynomial} \;=\; P_l(x) \;=\; \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \;\;,\;\; P_0(x)=1 \;\;,\;\; P_1(x)=x \;\;,\;\; P_2(x)=\frac{1}{2}(3x^2-1) \;\;,$$

$$\int_{-1}^{+1} dw P_\ell(w)P_{\ell'}(w)=\frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} \;=\; P^m_l(x) \;=\; (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} \;=\; Y^m_l(\theta,\;\phi) \;=\; (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P^m_l(\cos\theta)e^{im\phi} \;\;,$$

$$Y^0_0=\left(\frac{1}{4\pi}\right)^{1/2} \;\;,\;\; Y^0_1=\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \;\;,\;\; Y^{\pm1}_1=\mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y^0_2=\left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta-1) \;\;,\;\; Y^{\pm1}_2=\mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \;\;,\;\; Y^{\pm2}_2=\left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels :}\quad j_l(r) \;=(-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right) \;\;,\;\; n_l(r) \;=(-1)^{(\ell+1)}r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right) \;\;,$$

$$\text{with asymptotic behavior}\quad j_\ell(r)\to\frac{\cos(r-\ell\pi/2-\pi/2)}{r} \;\;,\;\; n_\ell(r)\to\frac{\sin(r-\ell\pi/2-\pi/2)}{r} \;.$$

$$j_0(r)=\frac{\sin r}{r} \;\;,\;\; n_0(r)=-\frac{\cos r}{r} \;\;,\;\; j_1(r)=\frac{\sin r}{r^2}-\frac{\cos r}{r} \;\;,\;\; n_1(r)=-\frac{\cos r}{r^2}-\frac{\sin r}{r} \;\;,$$

$$j_2(r)=\frac{3\sin r}{r^3}-\frac{\sin r}{r}-\frac{3\cos r}{r^2} \;\;,\;\; n_2(r)=-\frac{3\cos r}{r^3}+\frac{\cos r}{r}-\frac{3\sin r}{r^2} \;.$$

Problem 1

Consider a finite set of operators B_i. Let H be a Hamiltonian which *commutes* with each B_i; i.e., $[H, B_i] = 0$ for all i. Suppose the $|a_n\rangle$s form a complete set of eigenstates of H satisfying $H|a_n\rangle = a_n|a_n\rangle$.

(a) Let us choose one particular value of i and one particular value of n. Under what circumstances can it be deduced that $B_i|a_n\rangle$ is proportional to $|a_n\rangle$?

(b) Show that if the above is true for all i and for all n, then $[B_i, B_j] = 0$ for all i, j.

(c) How can you reconcile the rule stated in part (b) with the fact that for angular momentum operators L_i, we can have a situation where $[L_i, H] = 0$ but $[L_i, L_j] \neq 0$ when $i \neq j$?

Problem 2

Consider a quantum particle in 1D with mass m and energy $E = -E' < 0$ bound in the double δ-function potential $V(x) = -c_0\,\delta(x - L) - c_0\,\delta(x + L)$ where $c_0 > 0$.

(a) Derive the transcendental equation for the ground state energy E_0 and show (by plotting an appropriate freehand graph) that a solution of this equation exists, for all (positive) values of c_0.

(b) Derive the transcendental equation for the energy E_1 of the first (and only) excited state, show (by plotting an appropriate freehand graph) that it has a solution only if the constant c_0 is above a certain threshold $c_{\min}$, i.e. $c_0 > c_{\min}$, and determine the value of $c_{\min}$.

Problem 3

Let the 3D scattering of a quantum particle with mass m, energy $E > 0$, and $k = \sqrt{2mE}/\hbar$ off a spherically symmetric potential be described by the scattering amplitude $f(\theta)$ given by $f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\,P_l(\cos\theta)\,\sin\delta_l\,e^{i\delta_l}$. It is assumed that the potential falls off at large distances sufficiently fast.

(a) Using the expression for $f(\theta)$, derive the optical theorem connecting the total cross section σ with the imaginary part of the forward scattering amplitude.

(b) In s-wave scattering the differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = A$ is measured for a given (small) k. Determine the scattering amplitude $f(\theta)$. Is the result unique? What does "small k" mean in this context?

Problem 4

(a) Consider the angular momentum operator $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Evaluate the commutator $[L_x, az^2p_y^2 + bx^2r^2]$ where a and b are pure numbers and $r^2 = x^2 + y^2 + z^2$.

(b) Consider the addition of two angular momentum operators according to $\mathbf{L}_1 + \mathbf{L}_2 = \mathbf{L}$. Eigenstates $|\ell_1, m_1\rangle$ are associated with the operators $\mathbf{L}_1^2$ and L_{1z}, eigenstates $|\ell_2, m_2\rangle$ are associated with the operators $\mathbf{L}_2^2$ and L_{2z}, and eigenstates $|L, M\rangle$ are associated with the operators $\mathbf{L}^2$ and L_z. In terms of the quantum numbers (ℓ_1, m_1) and (ℓ_2, m_2) determine (i.e. derive as well as state the answer) the values which are allowed for the quantum numbers (L, M). Express the $|L, M\rangle$ eigenstate with the largest M value in terms of the $|\ell_1, m_1\rangle$ and $|\ell_2, m_2\rangle$ eigenstates.

(c) Consider a general ket $|\ell, m\rangle$ where ℓ designates the orbital angular momentum eigenvalue and m its z component. Consider a specific ket $|2, 1\rangle$. Determine for which $|\ell, m\rangle$ values the matrix elements

$$\langle 2, 1|r^2|\ell, m\rangle, \qquad \langle 2, 1|r\mathbf{r}|\ell, m\rangle$$

are non-zero, and determine their values (you can give your answers as closed form integrals in which everything is known with there being no need to actually evaluate the integrals).

Problem 5

A particle of mass M is constrained to move on the surface of a sphere of radius r. Its dynamics can be described by a free Hamiltonian H_0 and a set of free eigenstates $Y_\ell^m(\theta, \phi)$. The sphere is then embedded in a uniform gravitational field with acceleration g directed along the $-z$ axis, so that the particle experiences the potential

$$V(\theta, \phi) = mgr\cos\theta.$$

(a) Compute the values of all non-zero matrix elements of the potential operator $V(\theta, \phi)$ in a basis consisting of those $Y_\ell^m(\theta, \phi)$ that have $\ell = 0$, 1, and 2. Explicit forms for these spherical harmonics are given above. Hint: Do not attempt to work out every case, there are 81 of them! Use symmetry arguments to save work.

(b) Consider matrix elements of the full Hamiltonian between the states in this basis and the free ground state. Identify for which particular states in the basis these matrix elements are non-zero. Reduce the dimension of the basis to just these particular states, and write down the full Hamiltonian as a matrix in this reduced basis.

(c) Approximate the ground state of the full Hamiltonian as a superposition of the particular states found in part (b) and use a variational method to estimate the energy and wave function of the ground state of the full Hamiltonian in this basis. Hint: With the coefficients of the ground state wave function as the variational parameters, minimize the expectation value of the Hamiltonian.