**QUANTUM MECHANICS**

# Preliminary Examination

Friday 01/17/2014

09:00–13:00 in P-121

Answer a total of **FOUR** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or on sheets of paper stapled together. Make sure you clearly indicate who you are and what is the problem you are solving on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.

You are allowed to use a result stated in one part of a problem in the subsequent parts even if you cannot derive it. On the last page you will find some potentially useful formulas.

**Problem 1.** Let $H = H_{\text{kin}} + V(\vec{x})$ be a single-particle Hamilton operator with $H_{\text{kin}} = \vec{p}^{\,2}/(2M)$ and $M$ the mass of the particle. Consider the operator $D = \frac{1}{2}(\vec{x} \cdot \vec{p} + \vec{p} \cdot \vec{x})$.

(a) Calculate the commutators $[D, \vec{x}\,]$ and $[D, \vec{p}\,]$.

(b) Calculate $[D, F(\vec{x})\,]$ and $[D, G(\vec{p})\,]$ where $F$ and $G$ are differentiable functions. You may want to work out the commutators in position or momentum space.

(c) Let $H|E_i\rangle = E_i|E_i\rangle$. Calculate $[D, H\,]$ and prove that $2\langle E_i|H_{\text{kin}}|E_i\rangle = \langle E_i|\,\vec{x} \cdot \vec{\nabla} V(\vec{x})|E_i\rangle$, which is the quantum mechanical virial theorem.

**Problem 2.** Let us study the two-dimensional isotropic oscillator with the potential $U(x, y) = \frac{1}{2}\,m\omega^2(x^2 + y^2)$.

(a) Find the energy levels of the system with the quantum numbers of the $x$ and $y$ oscillators equal to $n_x$ and $n_y$. What are the degeneracies of the energy levels?

(b) Show that the Hamiltonian of the system commutes with (what would be the $z$ component of the three-dimensional) angular momentum

$$L = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

From the results of parts (a) and (b) we conclude that one can diagonalize both the Hamiltonian and the angular momentum operator with eigenfunctions $\psi_{n\ell}$, where $n = 0, 1, \ldots$ labels the solutions of the radial Schrödinger equation so that $E_{n\ell} = \hbar\omega(n + 1)$, and $\ell$ labels the angular states so that $L\psi_{n\ell} = \hbar\ell\psi_{n\ell}$. It turns out that for a given $n$, the value of $\ell$ runs from $-n$ to $n$ in steps of two.

(c) For the three lowest-energy levels $n = 0, 1, 2$ write explicitly the eigenfunctions $\psi_{n\ell}(x, y)$, which simultaneously diagonalize $H$ and $L$, in terms of the eigenfunctions of the one-dimensional harmonic oscillator states $\psi_{n_x}(x)$ and $\psi_{n_y}(y)$.

Recall that the unit-normalized lowest wave functions of the harmonic oscillator are of the form

$$\psi_0(x) = C\,e^{-\frac{\sigma^2 x^2}{2}},\ \psi_1(x) = C\sqrt{2}\,\sigma x\,e^{-\frac{\sigma^2 x^2}{2}},\ \psi_2(x) = C\sqrt{2}(\sigma^2 x^2 - \tfrac{1}{2})\,e^{-\frac{\sigma^2 x^2}{2}}$$

with $\sigma = \sqrt{m\omega/\hbar}$ and $C$ being an irrelevant common normalization constant.

**Problem 3.** Consider a two-level system with the Hamiltonian

$$\frac{H}{\hbar} = \lambda(|1\rangle \langle 2| + |2\rangle \langle 1|),$$

where $\lambda$ is real and $\{|1\rangle, |2\rangle\}$ makes an orthonormal basis in the corresponding Hilbert space.

(a) The system starts at $t = 0$ in the state $|\psi\rangle = C_1 |1\rangle + C_2 |2\rangle$. Show that at time $t$ the state is

$$|\psi(t)\rangle = C_1(\cos \lambda t \; |1\rangle - i \sin \lambda t \; |2\rangle) + C_2(\cos \lambda t \; |2\rangle - i \sin \lambda t \; |1\rangle) \, .$$

(b) In particular, suppose the system starts in state $|1\rangle$. What is the probability that it would be found in state $|2\rangle$ if a measurement were made at time $t$?

(c) Continuing from part (b), suppose that measurements are actually made at two times $t_1$ and $t_2$, with $t_2 > t_1$. What is the probability that the system is found in state $|2\rangle$ at both times?

**Problem 4.** (a) Show that if for some linear operator $Q$ the inner product equality $(\psi, Q\phi) = 0$ is valid for all vectors $\psi$ and $\phi$, then $Q$ is the zero operator that maps all vectors to the zero vector.

(b) Suppose every expectation value of an operator $Q$ is zero, $(\psi, Q\psi) = 0$ for all vectors $\psi$. Of course, then we also have $(\psi + \lambda\phi, Q(\psi + \lambda\phi)) = 0$, no matter what the vectors $\psi$ and $\phi$ and the scalar $\lambda$ are. By applying this observation with $\lambda = 1$ and $\lambda = i$, show that $(\psi, Q\phi) = 0$ for all $\psi$ and $\phi$. By combining with the result of part (a), we have the result $(\psi, Q\psi) = 0 \, \forall \psi \Rightarrow Q = 0$.

(c) Every operator $Q$ may be decomposed trivially in the form $Q = Q_1 + iQ_2$ with $Q_1 = \frac{1}{2}(Q + Q^\dagger)$ and $Q_2 = -\frac{i}{2}(Q - Q^\dagger)$. Show that $Q_1$ and $Q_2$ are hermitian.

(d) The expectation value of a hermitian operator in every state is real. Show that the reverse also holds true: An operator such that its expectation value in every state is real is necessarily hermitian. Combining the results of parts (b) and (c) is one possible way to proceed.

**Problem 5.** Suppose we know the normalized eigenstates $|n^{(0)}\rangle$ and the corresponding non-degenerate energies $E_n^{(0)}$ of the unperturbed Hamiltonian $H^{(0)}$. Let $V$ be a "small" time-independent perturbation.

(a) It is possible to use perturbation theory to write the eigenstates $|n\rangle$ and the corresponding energies of the Hamiltonian $H = H^{(0)} + V$ as series of the form $|n\rangle = |n^{(0)}\rangle + |n^{(1)}\rangle + \ldots$ and $E_n = E_n^{(0)} + E_n^{(1)} + \ldots$, where the quantities with the superscript $(j)$ scale with the $j^{\text{th}}$ power of the strength of the perturbation $V$. Show that the first- and second-order corrections to the energies are

$$E_n^{(1)} = \langle n^{(0)}|V|n^{(0)}\rangle \,, \qquad E_n^{(2)} = \sum_{m\neq n} \frac{|\langle n^{(0)}|V|m^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}} \,.$$

(b) Consider now the one-dimensional Hamilton operator $H^{(0)} = p^2/(2m) + \frac{1}{2}m^2\omega^2x^2$ and the perturbation $V = -Fx$. Using the formalism developed in part (a), calculate the eigenenergies of the Hamiltonian $H = H^{(0)} + V$ in first and second order in perturbation theory.

(c) Find the exact eigenenergies of the Hamiltonian $H = H^{(0)} + V$ of part (b), and check your perturbative results against them.

$$\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty$$

$$\int_{-\infty}^{+\infty} dx \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) \quad \text{with} \quad \mathrm{Re}(\alpha) > 0$$

$$\int_0^{\infty} dx\; x \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + i \frac{p}{m\omega}\right)$$