QUANTUM MECHANICS

Preliminary Examination

Friday 01/18/2013

9:00am - 1:00pm in P-121

Answer a total of **FOUR** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or on sheets of paper stapled together. Make sure you clearly indicate who you are and what is the problem you are solving on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.

On the last page you will find some potentially useful formulas.

Problem 1. An operator can be diagonalized if an orthonormal basis may be found whose members are eigenvectors of the operator. It is a built-in assumption in quantum mechanics that any hermitian operator may be diagonalized. Two hermitian operators may be diagonalized simultaneously (with the same orthonormal basis) if and only if they commute. On the other hand, an operator such that $NN^\dagger = N^\dagger N$, or $[N, N^\dagger] = 0$, is called *normal.*

(a) Every operator A may be decomposed trivially in the form $A = A_1 + iA_2$, where A_1 and A_2 are hermitian. Suppose we have a normal operator N with the corresponding components N_1 and N_2. Verify the following items: (i) $[N_1, N_2] = 0$. (ii) N may be diagonalized.

(b) Conversely, suppose that an operator N can be diagonalized, with the eigenvalues c_n (not necessarily real) and the orthonormal eigenvectors v_n. Verify the following items: (i) $(v_n, N^\dagger v_m) = c_n^*\,\delta_{nm}$. (ii) $N^\dagger v_m = c_m^* v_m$. Therefore, $N^\dagger$ can also be diagonalized, with eigenvalues and eigenvectors c_n^* and v_n. (iii) N is normal.

We have the result that normal, and only normal, operators can be diagonalized.

Problem 2. A simple harmonic oscillator in one dimension with mass m and frequency ω is driven by a small constant force F, which gives rise to an extra potential energy $H' = -Fx$.

(a) Find the energy eigenstates of the oscillator to the lowest nontrivial order in the perturbation strength F.

(b) While the expectation value of position is zero in any energy eigenstate of the unperturbed oscillator, the perturbation causes a displacement. Find the expectation value of position in the energy eigenstate $|n\rangle$ to the lowest nontrivial order in the perturbation.

Problem 3. A particle in one dimension interacts with a potential, $V(x) = -\alpha\delta(x)$ $(\alpha > 0)$.

(a) As is well known, such a delta function potential causes a discontinuity in the derivative of the wave function such that

$$\lim_{\epsilon\to 0} \psi'(0+\epsilon) - \psi'(0-\epsilon) = -\frac{2m\alpha}{\hbar^2}\psi(0).$$

Prove this.

(b) Find the ground state wave function and energy. Are there any excited bound states?

(c) Assuming a particle of momentum $\hbar k$ is incident from the left and scatters, what is the probability that it is reflected? Transmitted?

Problem 4. Find the Clebsch-Gordan coefficients for combining two spins with $s_1 = \frac{1}{2}$ and $s_2 = 1$.

Problem 5. In three dimensions the integral form of the Schrödinger equation takes the form

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}V(\mathbf{r}_0)\psi(\mathbf{r}_0)d^3\mathbf{r}_0,$$

where $\psi_0(\mathbf{r})$ is the solution to the free Schrödinger equation.

(a) In the limit of large $\mathbf{r}$ and for a particle scattering elastically off a localized potential $V(\mathbf{r}_0)$ (*i.e.*, $|\mathbf{r}_0| \ll |\mathbf{r}|$) show that the general expression for the scattering amplitude is

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2 A}\int e^{-i\mathbf{k}\cdot\mathbf{r}_0}V(\mathbf{r}_0)\psi(\mathbf{r}_0)d^3\mathbf{r}_0,$$

where $\psi_0(\mathbf{r}) = Ae^{ikz}$ is the incoming wave, and θ and ϕ are the polar and azimuthal scattering angles, respectively.

(b) Assuming a weak potential, take the expression from part (a) and show that the first Born approximation yields

$$f(\theta,\phi) \approx -\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}_0}V(\mathbf{r}_0)d^3\mathbf{r}_0,$$

where $\mathbf{k}' = k\hat{z}$.

(c) For the Yukawa potential,

$$V(\mathbf{r}_0) = \beta\frac{e^{-\mu r_0}}{r_0}$$

find the differential scattering cross-section. Notice that the potential is spherically symmetric. (Hint: let the polar axis of the $\mathbf{r}_0$ integral lie along $\boldsymbol{\kappa} = \mathbf{k} - \mathbf{k}'$).

(d) Use your result from (c) to obtain Rutherford's famous formula for Coulomb scattering.

$$\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty$$

$$\int_{-\infty}^{+\infty} dx \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) \quad \text{with} \quad \mathrm{Re}(\alpha) > 0$$

$$\int_0^{\infty} dx \; x \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + i\frac{p}{m\omega}\right)$$