Preliminary Exam: Quantum Physics 1/14/2011, 9:00-3:00

Answer a total of **SIX** questions of which at least **TWO** are from section **A**, and at least **THREE** are from section **B**. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. Some possibly useful information:

$$\nabla^2 \;=\; \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \;=\; \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\int_0^\infty dx\; x^n e^{-ax} = \frac{n!}{a^{n+1}}, \qquad \int_0^\infty dx\; e^{-a^2x^2} = \frac{\pi^{1/2}}{2a}, \qquad \int_0^\infty dx\; xe^{-a^2x^2} = \frac{1}{2a^2},$$

$$\text{Hermite polynomial} \;=\; H_n(x) \;=\; (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \quad , \quad H_0(x)=1 \quad , \quad H_1(x)=2x \quad , \quad H_2(x)=4x^2-2$$

$$\text{Laguerre} \;=\; L_n(r) \;=e^r\frac{d^n}{dr^n}\left(r^ne^{-r}\right) \quad , \quad \text{associated Laguerre} \;=\; L^q_{n+q}(r) \;=\; (-1)^q\frac{d^q}{dr^q}L_{n+q}(r) \quad .$$

$$\text{Legendre polynomial} \;=\; P_l(x) \;=\; \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \quad , \quad P_0(x)=1 \quad , \quad P_1(x)=x \quad , \quad P_2(x)=\frac{1}{2}(3x^2-1) \quad ,$$

$$\int_{-1}^{+1} dw P_\ell(w)P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} \;=\; P_l^m(x) \;=\; (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} \;=\; Y_l^m(\theta,\ \phi) \;=\; (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi} \quad ,$$

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \quad , \quad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \quad , \quad Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1) \;,\; Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \;,\; Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels :} \quad j_l(r) \;=(-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right) \quad , \quad n_l(r) \;=(-1)^{(\ell+1)}r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right) \quad ,$$

$$\text{with asymptotic behavior} \qquad j_\ell(r)\to\frac{\cos(r-\ell\pi/2-\pi/2)}{r} \quad , \quad n_\ell(r)\to\frac{\sin(r-\ell\pi/2-\pi/2)}{r} \quad .$$

$$j_0(r)=\frac{\sin r}{r} \quad , \quad n_0(r)=-\frac{\cos r}{r} \quad , \quad j_1(r)=\frac{\sin r}{r^2}-\frac{\cos r}{r} \quad , \quad n_1(r)=-\frac{\cos r}{r^2}-\frac{\sin r}{r} \quad ,$$

$$j_2(r)=\frac{3\sin r}{r^3}-\frac{\sin r}{r}-\frac{3\cos r}{r^2} \quad , \quad n_2(r)=-\frac{3\cos r}{r^3}+\frac{\cos r}{r}-\frac{3\sin r}{r^2} \quad .$$

Section A: Statistical Mechanics

A.1 An ideal gas of N atoms at temperature T is confined by an isotropic three-dimensional harmonic trap with potential energy

$$U(r) = \frac{m\omega^2 r^2}{2},$$

where m is the atom mass and ω is the characteristic trapping frequency.

(a) Calculate the system partition function.

(b) Show that the entropy of the trapped ideal gas is equal to

$$S = Nk\left[4 + \ln\left(\frac{(\hbar\omega)^3}{N(kT)^3}\right)\right].$$

(c) If the frequency ω is doubled adiabatically, calculate by how much the temperature changes.

(d) Show that density of the ideal gas in a trap is distributed as:

$$n(r) = N\left(\frac{m\omega^2}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m\omega^2 r^2}{2kT}\right).$$

A.2 The elementary excitations of a weakly interacting Bose-Einstein condensate in an ultralow-temperature dilute atomic gas are analogous to sound waves with dispersion relation

$$\epsilon(k) = \hbar c(k_x^2 + k_y^2 + k_z^2)^{1/2},$$

where c is the analog of the speed of sound.

(a) Calculate the average energy of the excitations.

(b) How does the heat capacity of the excitations depend on temperature?

A.3 In an ideal gas with N electrons the average number of particles occupying a single particle quantum state with energy ϵ is equal to

$$< n(\epsilon) >= \left[\exp\left(\frac{\epsilon - \mu}{kT}\right) + 1\right]^{-1}.$$

(a) Obtain a formula that can be used to determine the chemical potential μ in terms of the particle density $n = N/V$.

(b) What is the value μ_0 that μ takes at $T = 0^\circ K$?

(c) Show that the expression for the chemical potential reduces to the Boltzmann distribution in the limit $\lambda^3 n \ll 1$, where λ is the de Broglie wavelength:

$$\lambda = \hbar\left(\frac{2\pi}{mkT}\right)^{1/2}.$$

(d) Make a sketch of $< n(\epsilon) >$ versus ϵ at $T = 0^\circ K$ and at $T = \mu_0/10k$. Label significant points on both axes.

Section B: Quantum Mechanics

B.1 A qubit is a quantum analog of a classical information bit, a superposition of two states living in a two-dimensional Hilbert space $\mathcal{H}$ spanned by two vectors $|0\rangle$ and $|1\rangle$. Duplication of a qubit entails that there is another qubit that is initially in some normalized reference state $|e_0\rangle$ and that a transformation $U : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ exists so that $U|\psi\rangle \otimes |e_0\rangle = |\psi\rangle \otimes |\psi\rangle$ for any normalized $|\psi\rangle \in \mathcal{H}$. Show that if there is such a transformation in the first place, it cannot be unitary, and therefore cannot result from any Hermitian Hamiltonian acting on the two qubits. This is the no-cloning theorem of quantum information science.

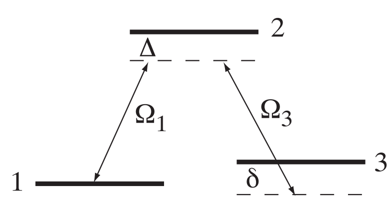
B.2 A three-level Λ scheme incorporates three states (say, in an atom) $|1\rangle$, $|2\rangle$ and $|3\rangle$ as in the figure below, and monochromatic light fields to drive the transitions $|1\rangle \to |2\rangle$ and $|3\rangle \to |2\rangle$. The laser fields are characterized by their Rabi frequencies Ω_1 and Ω_3, basically products of the electric field amplitude and the appropriate dipole moment matrix element. In this "rotating frame" the resonance conditions are governed by the "intermediate detuning" Δ and the "two-photon detuning" δ, whose roles as the amounts by which the energies of the photons undershoot the resonance conditions are also sketched in the figure. We consider the system on exact two-photon resonance, $\delta = 0$, whereupon the Hamiltonian reads

$$\frac{H}{\hbar} = \Delta|2\rangle\langle 2| + \Omega_1|2\rangle\langle 1| + \Omega_1^*|1\rangle\langle 2| + \Omega_3|2\rangle\langle 3| + \Omega_3^*|3\rangle\langle 2|\,.$$

(a) Show that the Hamiltonian has the eigenvalue $E = 0$, and find the corresponding normalized eigenvector. This is the so-called "dark state" $|D\rangle$. It has no component along the excited state $|2\rangle$.

(b) There is another normalized superposition of the states $|1\rangle$ and $|3\rangle$, the "bright state" $|B\rangle$, which is orthogonal to the dark state. Find the eigenvalues of the Hamiltonian in the subspace spanned by the vectors $|B\rangle$ and $|2\rangle$, and show that they are nonzero whenever $\Omega_1 \neq 0$ or $\Omega_3 \neq 0$.

(c) Suppose that the system starts in the state $|1\rangle$. The Rabi frequencies (laser field amplitudes) are varied so that initially $|\Omega_3| \gg |\Omega_1|$, and at the end $|\Omega_1| \gg |\Omega_3|$. Argue that if the variation of the Rabi frequencies is slow enough, the system ends up in the state $|3\rangle$ without ever visiting the excited intermediate state $|2\rangle$. This scheme is known as STIRAP.



B.3 Take what may be simplest possible model for a symmetric double-well potential in one dimension, namely $V(x) = V_0$ for $|x| < a$, $V(x) = 0$ for $a \le |x| \le b$, and $V(x) = \infty$ for $|x| > b$, with $V_0 > 0$ and $0 < a < b$.

(a) Show that the equations for the energies of even and odd bound states with $0 < E < V_0$ are, respectively,

$$\frac{\tan[(b-a)k]}{k} = -\frac{\coth a\kappa}{\kappa}, \quad \frac{\tan[(b-a)k]}{k} = -\frac{\tanh a\kappa}{\kappa},$$

where

$$k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}, \kappa = \left(\frac{2m(V_0 - E)}{\hbar^2}\right)^{1/2}.$$

(b) Show that in the limit $V_0 \to \infty$ the eigenstate energies are:

$$E_n = \frac{n^2\pi^2\hbar^2}{2m(b-a)^2}, \ n = 1, 2, \dots,$$

and that the states are doubly degenerate.

(c) Suppose now that V_0 is asymptotically large, much larger than any other relevant energy in the problem, but finite. For the purposes of the present argument we therefore assume that κ is a constant, equal to $\kappa = (2mV_0/\hbar^2)^{1/2}$. Each degenerate pair of energy eigenstates splits into a doublet with a small difference between the energies. Show that the splitting is approximately

$$\Delta E_n \simeq \frac{8E_n}{(b-a)\kappa} e^{-2\kappa a}.$$

(d) In case **(c)** which state in the doublet is lower in energy, even or odd?

B.4

For the trial wave function

$$\psi = (a_0)^{-3/2} \rho^\ell e^{-b\rho} Y_\ell^m(\theta\phi)$$

where $\rho = r/a_0$, $a_0 = \hbar^2/me^2$ use the variational method to calculate the lowest hydrogen atom energy level corresponding to angular momentum ℓ. For the hydrogen atom the Coulomb potential is $V(r) = -e^2/r$.

B.5

(a) For a point particle of momentum of magnitude k which is scattered by a potential $V(\vec{r})$ derive the first order Born approximation for the elastic scattering amplitude.

(b) Suppose the scattering potential is radially symmetric: $V = V(r)$. Show that the corresponding first order Born approximation for the scattering amplitude is given by

$$f_{\text{Born}}(\theta) = -\frac{2m}{\hbar^2 q} \int_0^\infty dr\, r\, V(r)\, \sin(qr)$$

where θ is the scattering angle and $q = 2k \sin(\theta/2)$.

(c) Consider scattering from a spherical potential of the form

$$V(r) = V_0 \quad , \quad r < a \quad ,$$

$$V(r) = 0 \quad , \quad r > a \quad .$$

Compute the first order Born approximation for both the differential and total cross sections.

Hint: The Helmholtz Green's function $G(\vec{x}, \vec{y})$, which satisfies $(\nabla^2 + k^2)G(\vec{x}, \vec{y}) = \delta^3(\vec{x} - \vec{y})$, is given by

$$G(\vec{x}, \vec{y}) = -\frac{e^{ik|\vec{x}-\vec{y}|}}{4\pi|\vec{x} - \vec{y}|}$$

B.6 Consider the addition of two angular momentum operators according to $\mathbf{L}_1 + \mathbf{L}_2 = \mathbf{L}$. Eigenstates $|\ell_1, m_1\rangle$ are associated with the operators $\mathbf{L}_1^2$ and L_{1z}, eigenstates $|\ell_2, m_2\rangle$ are associated with the operators $\mathbf{L}_2^2$ and L_{2z}, and eigenstates $|L, M\rangle$ are associated with the operators $\mathbf{L}^2$ and L_z.

(a) In terms of the quantum numbers (ℓ_1, m_1) and (ℓ_2, m_2) determine (i.e. derive as well as state the answer) the values which are allowed for the quantum numbers (L, M).

(b) In terms of the basis vectors $|\ell_1, m_1\rangle$ and $|\ell_2, m_2\rangle$ construct the particular eigenstates $|L, M\rangle$ which possess the three highest allowed positive M values. (Instead of solving this part for general ℓ_1, ℓ_2, for partial credit you can solve for $\ell_1 = 1$, $\ell_2 = 1$.)