Preliminary Exam: Quantum Physics 1/15/2010, 9:00-3:00

Answer a total of **SIX** questions of which at least **TWO** are from section **A**, and at least **THREE** are from section **B**. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. Some possibly useful information:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\int_0^\infty dx\ x^n e^{-ax} = \frac{n!}{a^{n+1}}, \quad \int_0^\infty dx\ e^{-a^2x^2} = \frac{\pi^{1/2}}{2a}, \quad \int_0^\infty dx\ x e^{-a^2x^2} = \frac{1}{2a^2},$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \ , \ H_0(x) = 1 \ , \ H_1(x) = 2x \ , \ H_2(x) = 4x^2 - 2$$

$$\text{Laguerre} = L_n(r) = e^r \frac{d^n}{dr^n}\left(r^n e^{-r}\right) \ , \ \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q \frac{d^q}{dr^q} L_{n+q}(r) \ .$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \ , \ P_0(x) = 1 \ , \ P_1(x) = x \ , \ P_2(x) = \frac{1}{2}(3x^2-1) \ ,$$

$$\int_{-1}^{+1} dw P_\ell(w) P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} = P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} = Y_l^m(\theta,\ \phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi} \ ,$$

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \ , \ Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \ , \ Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1) \ , \ Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \ , \ Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels:} \quad j_l(r) = (-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right) \ , \ n_l(r) = (-1)^{(\ell+1)} r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right) \ ,$$

$$\text{with asymptotic behavior} \quad j_\ell(r) \to \frac{\cos(r - \ell\pi/2 - \pi/2)}{r} \ , \ n_\ell(r) \to \frac{\sin(r - \ell\pi/2 - \pi/2)}{r} \ .$$

$$j_0(r) = \frac{\sin r}{r} \ , \ n_0(r) = -\frac{\cos r}{r} \ , \ j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r} \ , \ n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r} \ ,$$

$$j_2(r) = \frac{3\sin r}{r^3} - \frac{\sin r}{r} - \frac{3\cos r}{r^2} \ , \ n_2(r) = -\frac{3\cos r}{r^3} + \frac{\cos r}{r} - \frac{3\sin r}{r^2} \ .$$

Section A: Statistical Mechanics

A.1 Consider a one-dimensional closed chain of N spins s_i where $N \gg 1$. Each spin variable can take two values: $s_i = 1$ or $s_i = -1$. The spins interact with nearest-neighbor interactions of strength J, and they also interact with an external magnetic field h. The Hamiltonian of the system is

$$H[s] = -J \sum_{\{i,j\}} s_i s_j - h \sum_i s_i$$

The system is at temperature T.

(a) Show that the statistical sum for the system

$$Z = \sum_{\{s_i\}} e^{-\beta H[s]}$$

can be written as

$$Z = \mathrm{Tr}(\tau^N)$$

where τ is a 2×2 "transfer" matrix

$$\tau_{11} = \exp\{\beta J + \beta h\}; \qquad \tau_{12} = \tau_{21} = \exp\{-\beta J\}; \qquad \tau_{22} = \exp\{\beta J - \beta h\}.$$

(Hint: To do this write $e^{-\beta H[s]} = \Pi_{i=1}^{N} Q(s_i, s_{i+1})$ with $Q(s_i, s_{i+1}) = e^{\beta J s_i s_{i+1} + \beta \frac{h}{2}(s_i + s_{i+1})}$, and then map $e^{-\beta H}$ into the form above. Remember that the chain is closed, which means that $s_N = s_1$.)

(b) Evaluate the statistical sum Z by finding the eigenvalues of the matrix τ.

(c) Calculate the free energy $F(T, h)$ in the thermodynamic limit $N \gg 1$.

(d) Calculate the magnetization

$$m(h) = \frac{1}{N} \frac{\partial F}{\partial h} \; .$$

Show that at $h = 0$ the spontaneous magnetization $m(0)$ vanishes at all nonzero temperatures $T \neq 0$, while at exactly zero temperature $m(0) = 1$.

A.2 For an ideal gas of N non-interacting, non-relativistic particles of mass m in a volume V at a temperature T, one defines a partition function $Q_N(V, T)$ according to

$$Q_N(V, T) = \sum_{\{n_\mathbf{p}\}} g(\{n_\mathbf{p}\}) e^{-\beta E(\{n_\mathbf{p}\})}.$$

Here $\epsilon_\mathbf{p} = \mathbf{p}^2/2m$, $E(\{n_\mathbf{p}\}) = \sum_\mathbf{p} n_\mathbf{p} \epsilon_\mathbf{p}$, $g(\{n_\mathbf{p}\})$ is the number of states associated with each allowed occupation number configuration $\{n_\mathbf{p}\}$, and the summation is made over all allowed $\{n_\mathbf{p}\}$ subject to the constraint $N = \sum n_\mathbf{p}$. In addition one introduces a fugacity z and defines a grand partition function $\mathcal{Q}(z, V, T)$ according to

$$\mathcal{Q}(z, V, T) = \sum_{N=0}^{N=\infty} z^N Q_N(V, T).$$

(a) If the particles in the gas are spin one-half fermions, evaluate $\mathcal{Q}(z, V, T)$.

(b) For this $\mathcal{Q}(z, V, T)$ evaluate the average occupation number $\langle n_{\mathbf{p}} \rangle$.

(c) For the gas determine the pressure P and the energy density U/V. (You can reduce these expressions to one-dimensional integrals that you do not need to evaluate.) Then find a z-independent, closed form relation between U/V and P.

(d) What is the entropy of the gas? (Again, you can reduce the expression to a one-dimensional integral that you do not need to evaluate.)

A.3

(a) Taking into account the nucleons of number $A = N + Z$ and electrons of number Z in a given atom, which of the following atoms are composite bosons and thus could in principle form a Bose-Einstein condensate (BEC)? To receive credit, *explain your reasoning.*

(1) ^{1}H, A=1, Z=1
(2) ^{3}He, A=3, Z=2
(3) ^{4}He, A=4, Z=2
(4) ^{40}K, A=40, Z=19
(5) ^{40}Ca, A=40, Z=20
(6) ^{85}Rb, A=85, Z=37

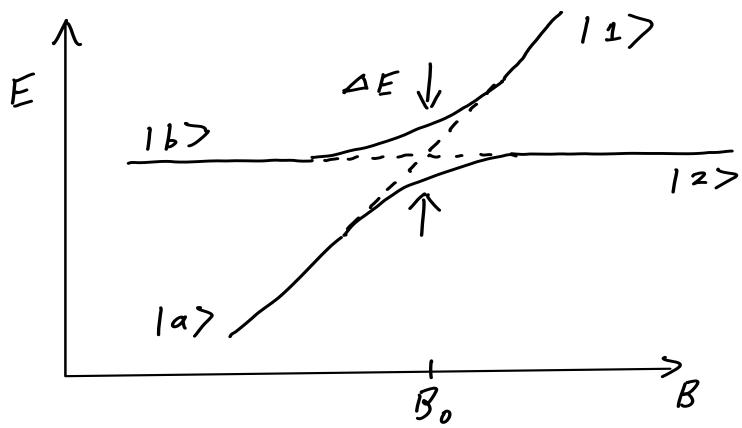
(b) For a gas of non-interacting bosonic atoms, a simple estimate of the BEC transition temperature T_c can be determined by solving for the temperature at which the de Broglie wavelength λ_{dB} is equal to the average interatomic spacing. Use this approach to estimate T_c for an atom of mass M in a gas with density $\mathcal{N}$ atoms per unit volume.

(c) In an optical lattice (which forms a periodic egg carton potential), things are more interesting because a condensate can form only if a sufficient fraction of the lattice sites are occupied. Assume that the lattice has N_s total sites, of which a fraction p are randomly occupied, so the total number of atoms is $N = pN_s$. The total number of possible states Ω of this lattice can then be found using simple combinatorics: it is the number of combinations of N_s objects taken N at a time. Find an expression for Ω, and use it to determine the entropy S associated with the partial filling of the lattice. Using Stirling's large n approximation $\ln n! = n\ln n - n$, simplify the result into the form $S = k_{\mathrm{B}} N_s f(p)$ assuming that both N_s and N are large.

(d) At a given temperature the lattice will undergo a BEC transition if S is greater than the corresponding entropy of a Bose-Einstein condensate, which is given by $S_{\mathrm{BEC}} = 1.283 k_{\mathrm{B}} N$ at $T = T_c$. Verify that at this temperature the BEC transition occurs at approximately $p = 0.54$ (Note: $\ln(0.54) = -0.616$ and $\ln(1 - 0.54) = -0.777$.)

Section B: Quantum Mechanics

B.1 In a time-dependent external magnetic field B a quantum system has two eigenstates $|a\rangle$ and $|b\rangle$. State $|a\rangle$ has an energy that depends linearly on B, which is ramped linearly in time so that the eigenenergy is $E_a = \alpha t$ where α is a constant. The other state, $|b\rangle$, has a constant energy E_b. A small perturbative coupling $\langle b|H'|a\rangle = V_{ab}$ is introduced, which as shown in the figure, leads to an avoided crossing at the field value B_0 where the energy levels would otherwise have intersected. In the presence of V_{ab}, a system that starts in state $|a\rangle$ and is swept *slowly* through the crossing as B is increased will follow the adiabatic trajectory shown by the solid line, ending up in the continuation of state $|b\rangle$ labeled as $|2\rangle$. However, if it is swept *rapidly*, it will instead take the so-called "diabatic" (i.e. non-adiabatic) path labeled as $|1\rangle$. Here we will estimate the probability that $|a\rangle$ takes the adiabatic path as it is swept through the crossing.



(a) If the coupling matrix element between the states $|a\rangle$ and $|b\rangle$ has a constant value $\langle b|H'|a\rangle = V_{ab}$, find the size ΔE of the avoided crossing at B_0.

(b) Now assume that in the presence of V_{ab} the system starts in state $|a\rangle$ and the magnetic field is ramped up linearly, so that $E_a = \alpha t$. Working in the $|a\rangle$, $|b\rangle$ basis set, use lowest-order time-dependent perturbation theory in H' to find the probability P_{2a} that the system ends up in the adiabatic final state $|2\rangle$. Because the linear ramp in energy is large, it cannot be treated perturbatively. However, it can be built into the zero-order wave function in the interaction representation by replacing the usual phase term with one having the time dependence:

$$e^{-iE_a t/\hbar} \rightarrow e^{-i\int_0^t E_a(t)dt/\hbar}.$$

Assume that the ramp extends well beyond the crossing on both sides, so that the limits of the integral given by perturbation theory can be extended indefinitely to $t = \pm\infty$. You will probably need to evaluate an integral that simplifies to the form:

$$\int_{-\infty}^{\infty} dt e^{-i(at^2+bt)} = (1-i)\sqrt{\frac{\pi}{2a}}\exp\left(\frac{ib^2}{4a}\right), \qquad \text{if } a > 0.$$

(c) A more exact non-perturbative approach gives the *Landau-Zener formula* for the adiabatic transition probability, which can be written for this case as

$$P_{2a}(\text{LZ}) = 1 - \exp\left(-\frac{2\pi V_{ab}^2}{\hbar\alpha}.\right)$$

Check your result by showing that it is equivalent to a lowest-order expansion of this expression for small probabilities.

B.2 Consider the Hamiltonian of a non-relativistic charged particle moving in the xy plane in a constant magnetic field B perpendicular to the plane:

$$H = \frac{1}{2m}\left[(p_x - \frac{e}{2}By)^2 + (p_y + \frac{e}{2}Bx)^2\right].$$

(a) Show (as is to be expected on physical grounds) that the Hamiltonian is invariant with respect to translations in the xy plane, and that the generators of the translations are

$$P_x = p_x + \frac{e}{2}By; \qquad P_y = p_y - \frac{e}{2}Bx.$$

(b) Calculate the commutation relation between the two generators P_x and P_y. This is called the "projective representation" of the translation group. Using this algebra show that even though P_x and P_y commute with the Hamiltonian H, one cannot diagonalize all three operators simultaneously. Prove that the ground state of the Hamiltonian H must be infinitely degenerate (i.e. that there is an infinite number of states with the same energy).

(c) Find an eigenstate of H which has a gaussian wave function of the form $\psi \propto \exp\left\{-(\Omega(x^2+y^2)/2\right\}$. Determine the associated frequency Ω, and calculate the energy of this state. By applying a translation transformation with parameters (a, b) to this state (i.e. translation a along x and b along y), find another state which is centered around the point $x = a,\ y = b$. Find the energy of this state.

(d) Now suppose that the plane is not infinite, but rather has a very large but finite area A. This can be achieved by placing a potential barrier along the boundary of a large region. In this situation translational symmetry strictly speaking is broken (it is not a symmetry anymore). Nevertheless, wave functions which are centered far away from the boundary practically do not feel the potential barrier and the energy splittings between them are exponentially small: $\Delta E \propto \exp\{-A\Omega\}$. Thus the ground state now is not infinitely degenerate but has finite degeneracy. Given the size of the eigenstates you found above, estimate the degeneracy when $A\Omega \gg 1$.

B.3

(a) A particle of mass m and energy $E = \hbar^2k^2/2m$ is incident from the left on the one-dimensional potential

$$V(x < -a) = 0, \qquad V(-a \le x \le 0) = -V_0, \qquad V(x > 0) = \infty,$$

where V_0 and a are positive constants. If the incoming wave is of the form of a plane wave $\psi(x) = Ae^{-ikx}$ where A is a constant, determine the form of the reflected wave function and its associated phase shift.

(b) Consider a three-dimensional Schrödinger equation with Hamiltonian

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(\mathbf{r}).$$

With the potential being time-independent, the Hamiltonian possesses a complete set of stationary eigenstates $\psi_j(\mathbf{r}, t) = \chi_j(\mathbf{r})e^{-i\omega_j t}$ with energies $E_j = \hbar\omega_j$. In terms of the Green's function $G(\mathbf{r}', t'; \mathbf{r}, t)$ any general wave function solution to the Schrödinger equation at time t' may be related to that at time t according to

$$\psi(\mathbf{r}', t') = i\int d^3r G(\mathbf{r}', t'; \mathbf{r}, t)\psi(\mathbf{r}, t).$$

Find a closed form expression for $G(\mathbf{r}', t'; \mathbf{r}, t)$ in terms of the eigenfunctions and eigenfrequencies of H.

B.4 The ground-state wave function of a non-relativistic, spinless hydrogen atom ($n = 1$, $\ell = 0$, $m = 0$) is

$$\psi_{1s}(r) = \frac{1}{\pi^{1/2} a_0^{3/2}} e^{-r/a_0}.$$

(a) The Hamiltonian describing a perturbation due to introducing an external electric field in the z-direction is

$$H' = e\mathcal{E}z = e\mathcal{E}r\cos\theta.$$

Show that this perturbation does not cause any first-order shift to the hydrogen atom ground-state energy.

(b) In second-order perturbation theory, the energy is shifted by coupling of the hydrogen atom ground state to the entire spectrum of $|np\rangle$ states via the matrix elements $\langle np|H'|1s\rangle$. Show that the energy shift can be written in the form $\Delta E = -(1/2)\alpha\mathcal{E}^2$, and write an expression for the polarizability α as an infinite sum over states, as expressed in terms of the unperturbed energies E_{1s} and E_{np} and the matrix elements $\langle np|H'|1s\rangle/\mathcal{E}$. Why do only the states with $\ell = 1$ contribute?

(c) Although an exact solution is possible, an approximate solution is much easier to find. Show that if it is assumed that all of the $|np\rangle$ states have approximately the same energy as the first excited state, i.e. $E_{np} \equiv E_{2p}$, then the infinite sum can easily be evaluated and the problem is reduced to evaluating the matrix element $\langle 1s|(H')^2|1s\rangle$. (Hint: Rearrange the sum over a product of matrix elements to take advantage of the completeness relation for a sum over a complete set of states.)

(d) Evaluate the integral for $\langle 1s|(H')^2|1s\rangle$ and use the explicit forms for the energy levels E_{1s} and E_{2p} to find a closed form expression for the polarizability α. Compare it with the exact value $\alpha = (9/2)a_0^3$ as given in cgs units. (Note that in cgs units the Rydberg constant can be expressed as $\mathrm{Ry} = e^2/2a_0$, where a_0 is the Bohr radius.)

B.5 Consider the two harmonic oscillator Hamiltonians

$$H_1 = \frac{p^2}{2m} + \frac{1}{2}\kappa^2 x^2, \qquad H_2 = \frac{(p-p_0)^2}{2\mu} + \frac{1}{2}\lambda^2(x-x_0)^2.$$

(a) What conditions should be satisfied by the coefficients $m,\ \mu,\ \kappa,\ \lambda,\ p_0$, and x_0 so that the two Hamiltonians are unitarily equivalent: i.e. that there exist a *unitary* operator U such that $U^\dagger H_1 U = H_2$.

(b) Given that the coefficients satisfy the required condition, construct the operator U.

(c) Consider a general homogeneous linear transformation

$$p \to \alpha p + \beta x; \qquad x \to \gamma x + \delta p$$

Find the conditions on the transformation coefficients $\alpha,\ \beta,\ \gamma,\ \delta$, in order for the transformation to be unitary.

B.6

(a) You are given N particles which have altogether N states available to them. How many distinct configurations are allowed for the system if the particles are (i) identical fermions, (ii) identical bosons? For each of these two cases write down the appropriate wave functions for each allowed configuration in the special case where $N = 3$.

(b) A single electron is prepared with its spin quantized in the positive z direction. The state is then rotated through an angle θ about the y axis.

(i) What is the electron spin state vector after the rotation?

(ii) Of what operator is the rotated state an eigenvector?

(iii) What is the eigenvalue associated with the eigenvector of part (ii)?