Preliminary Exam: Electromagnetism, Thursday August 25, 2016. 9:00-12:00

Answer a total of any **THREE** out of the four questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

1. Two concentric conducting spherical shells, the inner with radius a and outer with radius b, carry charges $+Q$ and $-Q$ respectively. The empty space between the shells is filled with a dielectric varying as $\epsilon = \epsilon_0 + \epsilon_1 \cos^2 \theta$ where θ is the polar angle and ϵ_0 and ϵ_1 are constants.

   (a) Write down an expression for the electrostatic potential in terms of Legendre polynomials, and hence find the electric field everywhere between the shells. In this problem the electric field is radial. Can you either prove or justify this?

   (b) Find the surface charge density on the inner shell.

   (c) Find the capacitance of the system.

2. (a) An insulator in the shape of a very long cylinder with radius R and volume charge density ρ spins with (angular) frequency ω around its axis. What is the magnetic induction **B** at a point on the axis?

   (b) How would your answer change if all the charge were concentrated on the surface?

   (c) How would your answer change if all the charge were not just concentrated on the surface, but is uniformly distributed on a line parallel to the axis of the cylinder? Is the associated magnetic induction time dependent or time independent.

3. A sphere of radius a and dielectric constant ϵ is placed in a uniform electric field **E**.
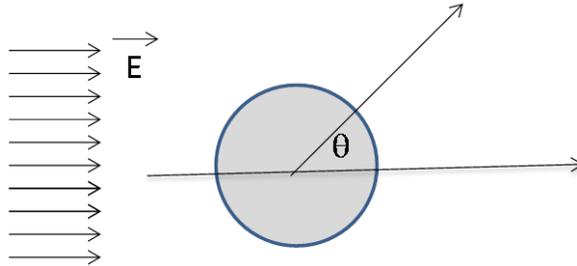

Figure 1: Sphere in an electric field

   (a) Determine the electric field inside the sphere.

   (b) Calculate the induced charge density on the sphere as a function of angle θ as shown in Figure (1). The sphere is embedded in empty spce with dielectric constant ϵ_0, and the electric field (shown) is asymptotically uniform outside the sphere.

4. A plane polarized electromagnetic wave traveling in a dielectric medium of real refractive index n is reflected at normal incidence from the surface of a conductor. The refractive index of the conductor is $n_2 = n(1 + ip)$ where p is real.

   (a) Find the ratio of the intensity of the reflected and incident waves.

   (b) Find the phase shift of the electric field of the reflected wave with respect to that of the incident wave.

Vector Formulas

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times \nabla\psi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla \cdot (\psi\mathbf{a}) = \mathbf{a} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{a}$$

$$\nabla \times (\psi\mathbf{a}) = \nabla\psi \times \mathbf{a} + \psi\nabla \times \mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla \cdot \mathbf{x} = 3 \qquad\qquad \nabla \times \mathbf{x} = 0$$

$$\nabla \cdot [\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \qquad \nabla \times [\mathbf{n}f(r)] = 0$$

$$(\mathbf{a} \cdot \nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a} \cdot \mathbf{n})] + \mathbf{n}(\mathbf{a} \cdot \mathbf{n})\frac{\partial f}{\partial r}$$

$$\nabla(\mathbf{x} \cdot \mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla \cdot \mathbf{a}) + i(\mathbf{L} \times \mathbf{a})$$

where $\mathbf{L} = \dfrac{1}{i}(\mathbf{x} \times \nabla)$ is the angular-momentum operator.

Theorems from Vector Calculus

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla \cdot \mathbf{A}\, d^3x = \int_S \mathbf{A} \cdot \mathbf{n}\, da \qquad \text{(Divergence theorem)}$$

$$\int_V \nabla \psi\, d^3x = \int_S \psi \mathbf{n}\, da$$

$$\int_V \nabla \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da$$

$$\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi)\, d^3x = \int_S \phi \mathbf{n} \cdot \nabla \psi\, da \qquad \text{(Green's first identity)}$$

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, d^3x = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n}\, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$

$$\int_S \mathbf{n} \times \nabla \psi\, da = \oint_C \psi\, d\mathbf{l}$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian
$(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical
(ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial \rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial \phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial \phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial \rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_2) - \frac{\partial A_1}{\partial \phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial\psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial \phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical
(r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial \theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial \phi}\right]$$
$$+ \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial \theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial\psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial \phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$