CLASSICAL MECHANICS/ ELECTRICITY AND MAGNETISM

Preliminary Examination

August 26, 2009

9:00 - 15:00 in P-121

Answer a total of **SIX** questions, choosing **THREE** from each section. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book. Make sure you clearly indicate who you are, and the problem you are answering. Double-check that you include everything you want graded, and nothing else.

SECTION A - Classical Mechanics

A1. A layer of dust with width h was formed over time by isotropic fall of meteorites onto the surface of the Earth (h is much smaller than the radius of the Earth). Show that the relative change in the length of the day due to the accretion of dust is approximately equal to $5hd/RD$ where R is the radius of the Earth, D is the density of the Earth and d is the density of dust.

Hint: Since the fall of meteorites is isotropic, the angular momentum of the Earth does not change.

A2. Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity of

$$v = \frac{v_1 + v_2}{1 + (v_1 v_2/c^2)},$$

which is the parallel-velocity addition law.

A3. A light, uniform, U-shaped tube is partially filled with mercury (total mass M, mass per unit length ρ) as shown in Figure 1. The tube is mounted so that it can rotate about one of the vertical legs. Neglecting friction, the mass and the moment of inertia of the glass tube, and the moment of inertia of the mercury column about the axis of rotation, do the following.

1. Calculate the potential energy of the mercury column and describe its possible motion when the tube is not spinning.

2. The tube is set in rotation with an initial angular velocity ω_0 with the mercury column at rest vertically with a displacement z_0 from the equilibrium.

   (b1) Obtain the Lagrangian for the system.

   (b2) Write down the equation of motion.

   (b3) What quantities are conserved in this motion? Give expressions for these quantities.
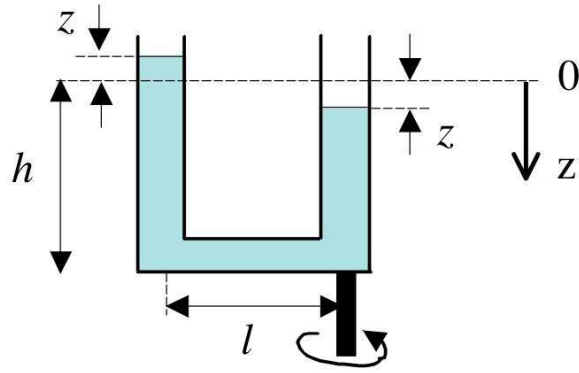
Figure 1: For problem **A3**.

A4. Consider the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 - \omega^2 x^2)e^{\gamma t}$$

for the motion of a particle of mass m in one dimension (x). The constants m, γ and ω are real and positive.

(a) Find the equation of motion.

(b) Interpret the equation of motion by stating the kinds of forces acting on the particle.

(c) Find the canonical momentum, and from this, construct the Hamiltonian. Is the Hamiltonian a constant of the motion? Is the energy conserved? Explain.

(d) For initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$, what is $x(t)$ asymptotically (i.e., as $t \to \infty$)?

SECTION B - Electricity and Magnetism

B1. (a) Show that the retarded electromagnetic potentials

$$\vec{A}(\vec{r},t) = \frac{1}{c} \int dV' \; \frac{\vec{j}(\vec{r}',t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|}; \qquad \phi(\vec{r},t) = \frac{1}{c} \int dV' \; \frac{\rho(\vec{r}',t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|}$$

satisfy the Lorenz condition

$$\vec{\nabla} \cdot \vec{A} - \frac{1}{c}\frac{\partial \phi}{\partial t} = 0.$$

Here $\vec{j}$ and ρ are the current density and the charge density respectively.

Hint: Use the fact that the electric current is conserved, namely

$$\frac{1}{c}\frac{\partial \rho(\vec{x},t)}{\partial t} - \frac{\partial j_i(\vec{x},t)}{\partial x_i} = 0.$$

(b) A point-like charge e moves along a straight line with constant velocity $\vec{v}$. Calculate the charge density and the current density due to the moving charge.

(c) Suppose that the charge moves slowly, such that $v \ll c$. In this case the time variation of the charge density and the current density are much slower than their spatial variation. Find the electric and magnetic fields of the moving charge in the quasistatic approximation, in which the time derivatives of ϕ and $\vec{j}$ are neglected.

B2. Consider a square wave-guide with side a, walls made of perfect conductors, and non-permeable medium with dielectric constant ϵ inside. Take the guide axis along the z-direction.

Starting from Maxwell's equations and assuming harmonic time-dependence of the fields, derive the wave equation for the field components.

Assuming solutions of the form $\vec{E}(\vec{r}) = \vec{E}(x,y)\exp{(ikz)}$ (and similarly for $\vec{B}$), solve the resulting 2-dimensional eigenvalue equation for the longitudinal (TM) eigenmodes of the guide.
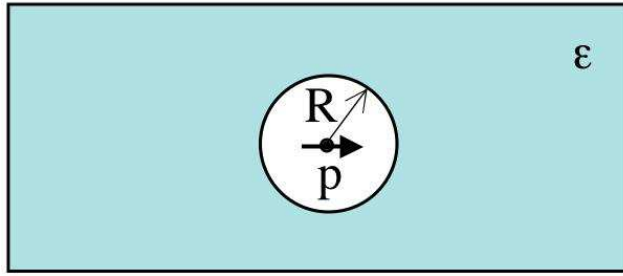
Figure 2: For problem **B3**.

Using Maxwell's equations, find the rest of the TM fields for this guide.

Hint: First find the transverse electric fields from E_z and then find $\vec{B}$ from $\vec{E}$.

What are the cutoff frequencies for the TM modes? Why are they called that?

B3. A spherical void of radius R is in otherwise homogeneous material of dielectric constant ϵ (see Figure 2). At the center of the void is a point dipole $\vec{p}$. Solve for the electric field everywhere.

B4. A plane-polarized electromagnetic wave of frequency ω is normally incident on a flat, non-permeable, material with dielectric constant ϵ and conductivity σ.

(a) Calculate the phase and amplitude of the reflected wave relative to the incident wave.

(b) Show that for a good conductor the reflection coefficient R is approximately given by

$$R \approx 1 - 2\frac{\omega}{c}\delta$$

where the skin depth $\delta = \sqrt{2/\sigma\omega}$.

(c) Discuss the limit of a very poor conductor.

5