Classical Mechanics/Electricity and Magnetism

Preliminary Exam

August 20, 2008

09:00 - 15:00 in P-121

Answer **THREE** (3) questions from each of the **TWO** (2) sections A and B for a total of **SIX** (6) solutions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented separately in an answer book, or on consecutively numbered sheets of paper stapled together. Make sure you clearly indicate who you are, and what is the problem you are solving. Double-check that you include everything you want graded, and nothing else.

Section A — mostly mechanics

A1. What is the gravitational force of a uniform sphere of mass M on an infinite uniform plane with the constant mass density (per unit area) σ? The sphere and the plane do not intersect.

A2. A fixed sphere A of radius a rests on a horizontal plane and a second sphere B of radius b ($b \leq a$) and mass m is placed upon it with the line from the center of sphere A to the center of sphere B at an angle θ_0 from the upward vertical. The second sphere then starts rolling down from rest in this position. The contact surface between the two spheres is rough.

Show that, as long as the spheres remain in contact and there is no sliding, the inclination of the line between the centers of the spheres from the upward vertical, θ, satisfies

$$7(a+b)\dot{\theta}^2 = 10g(\cos\theta_0 - \cos\theta)\,.$$

Moreover, show that the moving sphere will leave the fixed sphere when $\cos\theta = \frac{10}{17}\cos\theta_0$.

A3. Identify a set of principal axes and write down the inertia tensor for a supposedly uniform circular coin with respect to these principal axes. The precise values of the moments of inertia are unimportant here, but their relative magnitudes do matter.

The coin is tossed into the air with initial angular velocity components Ω_1 about a diameter through the coin and Ω_3 about an axis perpendicular to the coin through its center. Take

$$\vec{\omega}(t) = [\omega_1(t), \omega_2(t), \omega_3(t)]^T$$

to be the angular velocity vector at any time t with respect to the above axes during its motion.

(a) Show that if $\Omega_3 = 0$ and $\Omega_1 \neq 0$, then the coin simply spins about its diameter with angular velocity component $\omega_1(t) = \Omega_1$ throughout its motion.

(b) Explain how the angular velocity vector $\vec{\omega}$ evolves in time when both Ω_1 and Ω_3 are nonzero. Specifically, describe how $\vec{\omega}(t)$ changes in time t with respect to the principal axes.

A4. Recall that a constant (in time and space) magnetic field $\mathbf{B}$ may be derived from the vector potential $\mathbf{A}(\mathbf{r}) = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$.

(a) Consider a particle with mass m and charge q in the constant magnetic field $\mathbf{B} = B\,\hat{\mathbf{e}}_3$. The motion in the third direction is unaffected by the magnetic field, and is henceforth ignored. Show that the minimal coupling Hamiltonian for the charged particle reads

$$H = \frac{1}{2m}(p_1^2 + p_2^2) - \frac{\omega}{2}(x_1 p_2 - x_2 p_1) + \frac{m\omega^2}{8}(x_1^2 + x_2^2)\,.$$

Here x_1 and x_2 are the remaining Cartesian coordinates, p_1 and p_2 are the corresponding canonical momenta, and $\omega = qB/m$ is the cyclotron frequency.

(b) Show that the equations

$$\begin{aligned} x_1 &= \tfrac{1}{\sqrt{2m\omega}}\left[(\alpha_1 + \alpha_1^*) - i(\alpha_2 - \alpha_2^*)\right], & p_1 &= \tfrac{1}{2}\sqrt{\tfrac{m\omega}{2}}\left[-i(\alpha_1 - \alpha_1^*) - (\alpha_2 + \alpha_2^*)\right], \\ x_2 &= \tfrac{1}{\sqrt{2m\omega}}\left[-i(\alpha_1 - \alpha_1^*) + (\alpha_2 + \alpha_2^*)\right], & p_2 &= \tfrac{1}{2}\sqrt{\tfrac{m\omega}{2}}\left[-(\alpha_1 + \alpha_1^*) - i(\alpha_2 - \alpha_2^*)\right] \end{aligned}$$

define canonical-conjugate pairs $\{x_1, p_1\}$ and $\{x_2, p_2\}$ if [and, actually, only if] the new variables α_1 and α_2 have the Poisson brackets $[\alpha_1, \alpha_1^*] = [\alpha_2, \alpha_2^*] = -i$, and zero Poisson brackets for all other variable pairs picked from the set $\{\alpha_1, \alpha_1^*, \alpha_2, \alpha_2^*\}$.

(c) A direct substitution shows that, in terms of the variables α_1 and α_2, the Hamiltonian reads $H = \omega\alpha_1\alpha_1^*$. Show that the new variables evolve in time according to $\alpha_1(t) = \alpha_1(0)e^{-i\omega t}$, $\alpha_2(t) = \alpha_2(0)$.

(d) Describe the motion of the charged particle in physical terms.

Section B — mostly E&M

B1. No electrostatic trap.

(a) Consider any sphere with no charge inside. By applying a suitable Green's identity to the electrostatic potential $\phi(\mathbf{x})$ and the function $\psi(\mathbf{x}) = 1/|\mathbf{x}|$, show that the value of the potential at the center of the sphere equals the average of the potential over the surface of the sphere.

(b) Earnshaw's theorem says that an electrostatic potential cannot have a minimum or a maximum in free space. A purely electrostatic trap for a charged particle is therefore impossible. Prove Earnshaw's theorem on the basis of the result of part (a).

B2. A circular wire loop of radius a in the xy plane centered on the z axis carries the current $I(t)$. Assume that there is never a net charge in any macroscopic part of the wire loop. Also recall that in the Lorentz gauge, the manifestly causal retarded electromagnetic potentials from a distribution of charges and currents are

$$\phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\varrho\left[\mathbf{x}',t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right]}{|\mathbf{x}-\mathbf{x}'|}, \qquad \mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\mathbf{j}\left[\mathbf{x}',t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right]}{|\mathbf{x}-\mathbf{x}'|}\,.$$

(a) Verify the following expansions valid for $|\mathbf{x}| \gg |\mathbf{x}'|$:

$$|\mathbf{x}-\mathbf{x}'| = |\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|} + \mathcal{O}|\mathbf{x}'|^2, \qquad \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3} + \mathcal{O}|\mathbf{x}'|^2\,.$$

(b) Assume now that the current genuinely varies in time; $\mathbf{j}$, $\partial_{t'}\mathbf{j}$ and $\partial_{t't'}\mathbf{j}$ do not tend to zero with increasing distance from the origin $|\mathbf{x}|$ for any retarded time $t' = t - |\mathbf{x}|/c$ relevant to our argument. The leading contribution to the vector potential, in the limits when the current loop is small compared with the distance to the point of observation and the observation point is "sufficiently far" away, reads

$$\mathbf{A}(\mathbf{x},t) \simeq \frac{\mu_0}{4\pi c|\mathbf{x}|^2}\int d^3x'\,\mathbf{x}'\cdot\mathbf{x}\,\left.\frac{\partial\mathbf{j}(\mathbf{x}',t')}{\partial t'}\right|_{t'=t-|\mathbf{x}|/c} = \frac{\mu_0 a^2}{4c|\mathbf{x}|^2}\frac{d}{dt}I(t-|\mathbf{x}|/c)\;\hat{\mathbf{e}}_z\times\mathbf{x}\,.$$

Fill in the details to verify these assertions.

(c) What is the electric field far from the current loop? How does it fall off with increasing distance?

B3. Find the magnetic field due to a uniformly magnetized, spherical permanent magnet everywhere in space. Discuss the difference between magnetic induction and magnetic field using your solution. Take the permeability of the surrounding material to be μ_0, and the magnetization of the sphere to be $M_0\hat{\mathbf{z}}$.

B4. An electromagnetic plane wave of (angular) frequency ω is propagating through an optically active (chiral) medium so that the polarization of the medium is $\mathbf{P} = (2\gamma/c\mu_0\omega)\nabla\times\mathbf{E}$, with $\gamma \ll 1$. The medium has no free charges or currents, and it will not magnetize.

(a) In the absence of free charges one of Maxwell's equations for a polarizable medium says $\nabla\cdot\mathbf{D} = 0$, where the electric displacement is $\mathbf{D} = \epsilon_0\mathbf{E}+\mathbf{P}$ as usual. Show that in the present case $\nabla\cdot\mathbf{E} = 0$ holds true, too.

(b) Where convenient you may, of course, assume that the wave with the real wave number k propagates along the z axis. By virtue of $\nabla\cdot\mathbf{E} = 0$, the complex amplitude vector of the electric field is then of the form $\mathcal{E}_x\hat{\mathbf{e}}_x+\mathcal{E}_y\hat{\mathbf{e}}_y$. Show that the amplitudes $\mathcal{E}_x$ and $\mathcal{E}_y$, the wave number k, and the frequency ω have to satisfy the equations

$$\begin{bmatrix} k^2-k_0^2 & 2i\gamma k_0 k \\ -2i\gamma k_0 k & k^2-k_0^2 \end{bmatrix}\begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{bmatrix} = 0\,,$$

where $k_0 = \omega/c = 2\pi/\lambda$ would be the wave number in vacuum.

(c) There will be two plane wave modes of electromagnetic fields in the medium with different refractive indices. Find the refractive indices and characterize the modes at least to the lowest nontrivial order in γ.