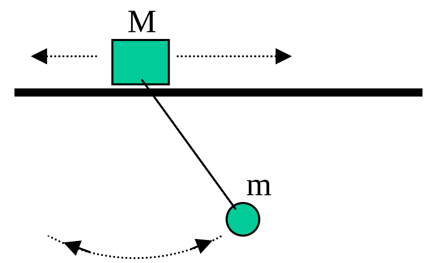
Classical Mechanics / Electricity and Magnetism

General Exam Questions for August 22, 2007

Instructions

Answer three questions from each of the two sections, for a total of six problems. Put each of your solutions in a separate answer book. Make sure that you label and sign your name on the cover of each book.

I. Classical Mechanics



1. Consider the system shown in the figure at the right. The mass M can move freely without friction along the x-direction on a fixed surface. Another mass m is attached to the first mass through a massless rod of length ℓ that pivots such that it swings freely in the vertical xy plane.
   a) How many degrees of freedom does the system have? Write down its Lagrangian in terms of those degrees of freedom you have identified.
   b) What is the equilibrium condition for the system?
   c) What are the normal modes for small deviations from the equilibrium? What are their frequencies?

2. It is well known that a uniform and constant magnetic field $\vec{B}$ may be derived from the vector potential $\vec{A} = \frac{1}{2}(\vec{B} \times \vec{r})$. What may be less well known is that the same expression for $\vec{A}$ also describes the situation where a uniform magnetic field is changing with time. Consider a particle of mass m and charge q that is constrained to move along a circle of radius ℓ in the xy plane, subject to the magnetic field $\vec{B} = B(t)\hat{k}$.
   a) What is the Lagrangian for this system?
   b) Find the equation of motion for the rotation angle of the particle along the circle using Lagrangian mechanics.
   c) There is a major qualitative difference in the motion depending on whether the magnetic field depends on time or not. Explain in physical terms what is going on.

3. ^{7}Li is bombarded with deuterons of $E = 10$ MeV energy, yielding ^{8}Be and a neutron. The reaction is exothermic with an energy yield of $E' = 14.5$ MeV. In this problem just treat a nucleus with N nucleons as having mass Nm_0, where $m_0 = 940$ MeV$/c^2$. You may use nonrelativisitic kinematics if you can justify doing so.
   a) What is the energy of the final-state neutron in the center of mass frame of the reaction?
   b) Find the energy of the emergent neutron in the laboratory coordinate system as a function of the angle between its direction of emergence and the direction of the

incident deuteron, both in the laboratory system. You may write your answer in parametric form.

c) What is the scale of the error in your result that comes from treating the problem non-relativistically?

4. The lock of a car door will only engage if the angular velocity of the closing door exceeds some threshold value ω. The door swings without friction about vertical hinges and has a radius of gyration k about a vertical axis through the hinges. The center of gravity of the door is a distance a from the hinge axis. The car is initially at rest with the door open at right angles to the side of the car. The car then accelerates forward in a straight line at constant acceleration f.

   a) Write down Newton's second law for rotation of a rigid body about an axis fixed to a point on the body, in an inertial reference frame.

   b) Show how to modify the equation of motion in part (a) to cover the case of a uniformly accelerating frame. Explain why this is justified.

   c) Show that the door will not close unassisted unless $f > \omega^2 k^2/(2a)$.

II. Electricity and Magnetism

1. A point charge q is fixed at a distance D from a conducting sphere of radius R and net charge Q.

   a) What is the force of attraction or repulsion between the point charge and the sphere?

   b) Show that this force is attractive when Q and q are of the same sign if and only if

$$\frac{Q}{q} < \frac{RD^3}{(D^2 - R^2)^2} - \frac{R}{D}$$

2. The half-space $z < 0$ is empty and the half-space $z > 0$ is filled with an insulating medium with a position-dependent dielectric constant $\epsilon(z)$. In the region $z < 0$ there is a constant uniform electric field $\vec{E}_0$.

   a) What is the electric field $\vec{E}(z > 0)$ if $\vec{E}_0$ is perpendicular to the $z = 0$ plane?

   b) What is the electric field $\vec{E}(z > 0)$ if $\vec{E}_0$ lies within the $z = 0$ plane?

3. Consider a conducting spherical shell of radius a. Outside the shell, the potential is

$$\phi = -E_0\, r \cos\theta \left(1 - \frac{a^3}{r^3}\right)$$

   where E_0 is a constant, $r^2 = x^2 + y^2 + z^2$, and θ is the polar angle between the vector $\vec{r}$ and the z-axis.

   a) This potential is created by a static background electric field. What is the electric field outside the shell, and what is the applied field? (*Hint: Very far away from the sphere, the influence of the sphere should be vanishing.*)

   b) Show that the surface charge density on the shell is given by

$$\sigma = 3\epsilon_0 E_0 \cos\theta$$

   c) Show that the force on a small piece of the shell normal to the surface is

$$d\vec{F} = \frac{\sigma^2}{2\epsilon_0} d\vec{S}$$

where σ is any surface charge density and $d\vec{S}$ is the surface element pointing outward. (*Hint: Remember that very close to the surface of the sphere its surface appears like a plane.*)

d) If one were to cut the shell in the middle, perpendicular to the z-axis, what would be the force one would have to use on each half-shell to keep the halves from flying apart?

4. A coaxial cable of inner radius a and outer radius b is filled in the region between the conductors with a dielectric with dielectric constants ϵ and μ. The cable supports a TEM mode whose complex electric field is of the form

$$\vec{E}(\vec{x}, t) = -\frac{\Phi}{\rho} e^{i(kz-\omega t)} \hat{e}_\rho$$

where Φ is a constant that characterizes the amplitude of the electromagnetic fields, real for the sake of argument, and $k = \omega\sqrt{\mu\epsilon}$.

a) Show that the corresponding magnetic field must be

$$\vec{B}(\vec{x}, t) = -\frac{\Phi}{\rho}\sqrt{\mu\epsilon}\, e^{i(kz-\omega t)} \hat{e}_\phi.$$

b) Boundary conditions on perfectly conducting metal give the surface current density. Show that the physical currents that pass through an orthogonal cross section of the cable at z in the outer shell and in the inner wire are

$$I(t) = \pm 2\pi\Phi\sqrt{\frac{\epsilon}{\mu}}\cos\left(kz - \omega t\right).$$

c) If the transverse dimensions of the cable are much smaller that the wavelength of the radiation at frequency ω, it makes sense to define the instantaneous physical potential difference between the inner and outer and inner conductors $V(t)$ as if the electric field were static. Find $V(t)$.

d) While is is possible to put an arbitrary dc voltage and current through a coaxial conductor, perhaps surprisingly, the quotient of the voltage to current (impedance) is fixed in the ac case. What is the impedance of the coaxial cable?