CLASSICAL MECHANICS/ ELECTRICITY AND MAGNETISM

Preliminary Examination

January 12, 2011

09:00 - 15:00 in P-121

Answer a total of **SIX** questions, choosing **THREE** from each section. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book, or on sheets of paper stapled together. Make sure you clearly indicate who you are, and the problem you are answering on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.

Page 5 contains the formulas for standard vector operators in the Cartesian, cylindrical, and spherical coordinates.

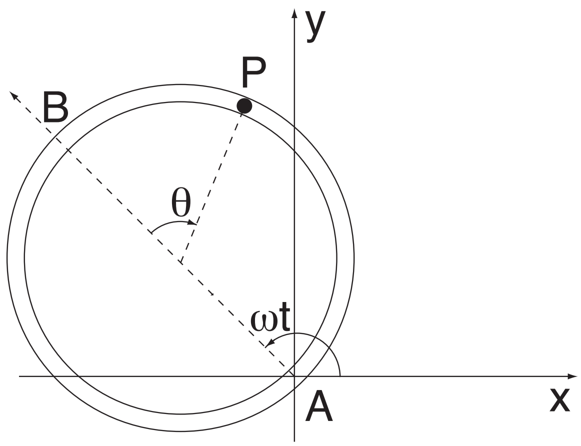

Figure 1: For problem **CM2**.

SECTION CM - Classical Mechanics

CM1. Consider the Lagrangian

$$L = \frac{m}{2}(\dot{x}^2 - \omega^2 x^2)e^{\gamma t}.$$

The motion of a particle of mass m is in one dimension. The constant m, γ, and ω are real and positive.

(a) Find the equation of motion.

(b) Physically interpret the equation of motion by stating the general nature of the forces, to which the particle is subject.

(c) Find the canonical momentum and from this construct the Hamiltonian function.

(d) Is the Hamiltonian a constant of motion? Is the energy conserved? Explain.

(e) For the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$, what is $x(t)$ asymptotically as $t \to \infty$?

CM2. A smooth, hollow, circular tube of radius R is fixed in a horizontal plane at one point A and contains a particle P of mass m. The particle is initially at rest at the opposite end B of the diameter through A. The tube is then made to rotate, with constant angular velocity ω about the vertical axis through A. Denote by θ the angle subtended at the center of the circular tube by BP.

(a) Construct the Lagrangian L of the particle as a function of the coordinate $\theta(t)$ and velocity $\dot{\theta}(t)$.

(b) Obtain Lagrange's equation of motion for the $\theta(t)$ coordinate and show that $\ddot{\theta} + \omega^2 \sin\theta = 0$.

(c) Show that $E = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L$ is the integral of motion and determine the value of this integral using the given initial conditions.

(d) Find the angle $\theta(t)$. What is $\theta(t)$ asymptotically as $t \to \infty$?

Hint: The problem may be solved both in the laboratory and rotating frames and both methods yield the same differential equation for $\theta(t)$. Be careful with the initial conditions.

CM3. A manned spacecraft is traveling at a uniform speed of $3c/5$ (c being the speed of light in a vacuum) along a straight line towards a distant planet 12 light years away from the Earth. Suppose that the journey began from the Earth at $t = 0$ (Earth's clock) and $t' = 0$ (spacecraft's clock).

(a) How long would it take for the spacecraft to reach the planet as measured on the Earth's frame and as measured in the spacecraft. If you get different values, explain why.

(b) When the spacecraft reaches the planet, what does the clock on the Earth read, according to the astronaut?

CM4. Three interacting particles with equal masses m are involved in one-dimensional motion along the x-axis. Particles 1 and 2 and particles 2 and 3 are connected to each other by identical ideal springs of constant κ (particle 2 is always in a central position). Construct Lagrange's equations, describing the motion of particles in the Center of Mass frame, and calculate the eigenfrequencies of vibration.

SECTION E&M - Electricity and Magnetism

E&M1. Consider electromagnetic waves in a wave guide, propagating along the z-direction. Write down possible forms for the electric and magnetic fields $\mathbf{E}(x, y, z, t)$ and $\mathbf{B}(x, y, z, t)$ for such waves. Using Maxwell's equations, show that TEM waves (where E_z and B_z are both zero) cannot occur in a completely hollow wave guide.

E&M2. An infinite plane slab of thickness a is charged to a volume density $\rho(x)$:

$$\rho(x) = \rho_0 \, \frac{x}{a}, \text{ if } |x| \leq a/2,$$

$$\rho(x) = 0, \qquad \text{if } |x| > a/2,$$

where ρ_0 is a positive constant and the x-axis is perpendicular to the slab surface. The origin of coordinates $x = 0$ is placed into the center of the slab. Find the potential $\phi(x)$ and electric field $\mathbf{E}(x)$ inside and outside the slab.

E&M3. A sphere of radius a has a bound charge Q distributed uniformly over its surface. The sphere is surrounded by a uniform fluid medium with the dielectric constant ϵ_r. The fluid also contains a free charge density given by

$$\rho(r) = -k\phi(r),$$

where r is the distance from the center of the sphere, k is a positive constant, and $\phi(r)$ is the electric potential at r. Using the Poisson equation find the potential everywhere letting $\phi(r) = 0$ as $r \to \infty$.

E&M4. A constant electric current flows through an infinite cylindrical non-magnetic conductor of radius a. The axially symmetric vector of current density $\mathbf{j}(\rho)$ depends on the distance ρ from the cylinder z-axis:

$$\mathbf{j}(\rho) = j_0(a/\rho)\mathbf{e}_z,$$

where j_0 is a positive constant and $\mathbf{e}_z$ is a unit vector of z-direction. Determine the magnetic vector potential $\mathbf{A}(\rho)$ inside and outside this conductor.

Standard Vector Operations in Common Coordinate Systems

Cartesian coordinates x, y, z

$$
\begin{aligned}
\nabla &= \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{e}}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{e}}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}
$$

Cylindrical coordinates ρ, ϕ, z

$$
\begin{aligned}
\nabla &= \hat{\mathbf{e}}_\rho \frac{\partial}{\partial \rho} + \hat{\mathbf{e}}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\
\nabla \cdot \mathbf{A} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_\rho \left[\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] + \hat{\mathbf{e}}_\phi \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] + \hat{\mathbf{e}}_z \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right] \\
\nabla^2 &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}
$$

Spherical coordinates r, θ, ϕ

$$
\begin{aligned}
\nabla &= \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \\
\nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta A_\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_r \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta} (\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi} \right] + \hat{\mathbf{e}}_\theta \left[\frac{1}{r \sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right] + \hat{\mathbf{e}}_\phi \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \\
\nabla^2 &= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2} \\
&\left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \equiv \frac{1}{r} \frac{\partial^2}{\partial r^2} r \right]
\end{aligned}
$$