Classical Mechanics

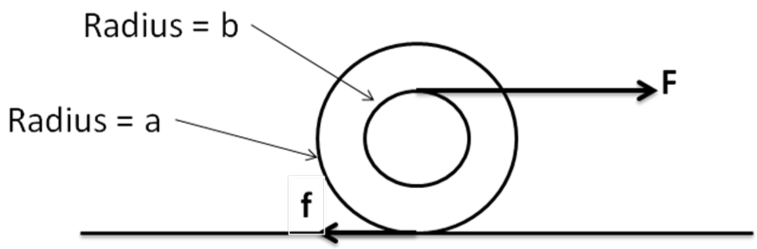
Preliminary Examination
Monday, January 11, 2016 ; 9.00 AM to 12.00 Noon

Answer a total of any THREE out of the four problems. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

Problem 1:

A wheel (shown in the figure) of mass m, and a moment of inertia $I = mR^2$ (R is a constant with the dimension of length) is pulled along a horizontal surface by the application of a horizontal force F to a rope unwinding from an axle of radius b as shown in the Figure. There is a friction force (f) between the wheel and the surface such that the wheel rolls without slipping.

(a) What is the acceleration of the wheel?
(b) Determine the friction force (f).



Problem 2:

A rocket with mass M (without fuel) is loaded with fuel of mass m_0. It takes off vertically in a uniform gravitational field (acceleration due to gravity = g). It ejects fuel at a velocity u with respect to the rocket. The fuel is completely ejected in a time T.

(a) Determine the equation of motion for the rocket
(b) Find a general solution of the equation of motion
(c) Calculate the velocity of the rocket at the instant T when all the fuel is ejected.

Problem 3:

See figure: Two masses, m_1 and m_2 are connected by a massless, inextensible cord of length ℓ that passes through a hole in a horizontal table. Mass m_1 moves without friction on the table top, in a curved path as shown, while mass m_2 oscillates like the bob of a pendulum. Gravity is acting downward. Use, e.g., the generalized coordinates suggested in the figure to solve this problem.

a) Set up the Lagrangian for this system. How many degrees of freedom are there?

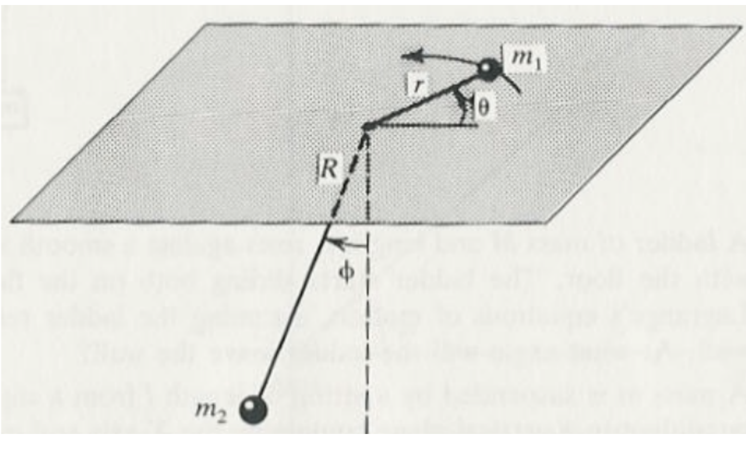
b) What is the constraint? Write the equation of constraint. What is the physical significance of the constraint force?

c) Assume now that mass m_2 only moves in one plane, as indicated in the figure. Use Lagrange's equations along with the method of Lagrange multipliers to derive the equations of motion.

d) What are the constants of motion?

e) Using the results of the previous parts along with the equation of constraint, go as far as you can towards finding the force of constraint and solving the equations of motion.

f) If there is an additional constraint that mass m_1 moves in a circle of radius $r = a$, show that the equations of motion decouple into the usual equations for a particle (m_1) moving in a circle and for a simple pendulum (m_2).



Problem 4:

A particle orbit is given by

$$r(\theta) = A\, e^{B\theta} \ ,$$

where A and B are positive constants. This is a logarithmic spiral.

a) Show that the force $F(r) = C/r^3$ (what is C?) and calculate the corresponding potential energy $V(r)$.

b) For angular momentum L calculate the effective potential $V_{eff}(r)$. Make a qualitative plot of $V_{eff}(r)$ vs. r for different L. Qualitatively discuss the orbit of the particle for different energies E.

c) Calculate $r(t)$ and $\theta(t)$ for the particle orbit.