Particle Number Fractionalization of an Atomic Fermi-Dirac Gas in an Optical Lattice

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We show that a dilute two-species gas of Fermi-Dirac alkali-metal atoms in a periodic optical lattice may exhibit fractionalization of the particle number when the two components are coupled via a coherent electromagnetic field with a topologically nontrivial phase profile. This results in fractional eigenvalues of the spin operator with vanishing fluctuations. The fractional part can be accurately controlled by modifying the effective detuning of the electromagnetic field.

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Particle number fractionalization is a remarkable phenomenon in both relativistic quantum field theory and condensed matter systems [1,2]. Jackiw and Rebbi [3,4] showed that, for a fermionic field coupled to a bosonic field with a topologically nontrivial soliton profile, the fermion number can be fractional. The noninteger particle number eigenvalues may be understood in terms of the deformations of the Dirac sea (or the hole sea) due to its interaction with the topologically nontrivial environment. In this paper we propose a manifestation of this phenomenon in the atomic regime, using an optically trapped Fermi-Dirac (FD) atomic gas [5]. A fractional fermion number has been demonstrated previously in the condensed matter regime in 1D conjugated polymers by Su, Schrieffer, and Heeger [4,6,7]. Fractionally charged excitations are also fundamental to the fractional quantum Hall effect [8], but the fractionalization mechanism is very different from that in the polymers and in the atomic gas in this paper. Our dilute atomic gas has a possible advantage, compared to condensed matter systems, in the sense that the interatomic interactions are weak and well understood, and there now exists a wide range of atomic physics technology to detect, manipulate, and control atoms by means of electromagnetic (em) fields.

We study a two-species atomic FD gas in a 1D optical lattice, coupled to an em field with topological properties similar to a soliton, or a phase kink. We show that this topologically nontrivial coherent background field results in fractionalization of the particle number operator eigenvalues for the fermionic atoms. Also, the spin operator has fractional eigenvalues with vanishing fluctuations.

The background field is generated by means of a coherent em field inducing transitions between the two fermionic components occupying different internal levels, e.g., by using experimentally realized technology of rapidly rotating laser beams [9,10]. In the low-energy limit we demonstrate the one-to-one correspondence of the Hamiltonian for the FD atoms to the Jackiw-Rebbi relativistic Dirac Hamiltonian describing fractionalization in quantum field theory. This is related to fractionalization in the polyacetylene polymer systems, where the linearized lattice vibrations are coupled to the electron dynamics which becomes analogous to that for a relativistic Dirac equation exhibiting a fractional particle number. We also show how our proposed system could be generalized to higher spatial dimensions to represent the fractionalization in relativistic $(2+1)$D and $(3+1)$D quantum field theories.

Tremendous progress in experiments on cold trapped atomic gases has allowed numerous studies with Bose-Einstein condensates and the cooling of a FD gas to the quantum degenerate regime [11]. Recently, condensates have been loaded into a periodic optical lattice [12]. Similar experimental progress is anticipated for FD gases.

The vacuum state in an atomic condensate can support topological excitations including defects, such as solitons [9] and vortices [10], as well as SU(2) Skyrmion textures [13]. In this paper we show that in atomic gases also the vacuum state *itself* may display nontrivial topological quantum numbers with a strong analogy to the vacua encountered in relativistic quantum field theories.

The phenomenon of a fractional fermion number is best illustrated by the following $(1+1)$D Dirac Hamiltonian density [3,4] for a two-component spinor $\Psi(x)$ coupled to a bosonic condensate $\varphi(x)$, which can be taken to be a static classical background field:

$$\mathcal{H} = c\hbar\Psi^\dagger\sigma^2\frac{d\Psi}{idx} + \hbar g\varphi\Psi^\dagger\sigma^1\Psi + mc^2\Psi^\dagger\sigma^3\Psi\,. \tag{1}$$

Here σ^i denote the Pauli spin matrices, g is the coupling coefficient, m is the fermionic mass, and c is the velocity of light. We show below that the atomic FD Hamiltonian can be written in this form.

We assume the bosonic field has a doubly degenerate ground state with constant field $\varphi(x) = \pm\gamma$. The vacuum then exhibits a spontaneously broken reflection symmetry $\varphi \leftrightarrow -\varphi$. For $m = 0$, the Dirac Hamiltonian has a charge conjugation symmetry, so that for every eigenvalue ϵ there exists an eigenvalue $-\epsilon$, and the corresponding eigenfunctions are paired according to $\Psi_{-\epsilon} = \sigma^3\Psi_\epsilon^*$. The fermion particle number operator is

$$N \equiv \frac{1}{2}\int dx\,[\Psi^\dagger(x), \Psi(x)]\,. \tag{2}$$

In the free vacuum, the fermion particle number vanishes.

The soliton background $\varphi(x)$ interpolates between the two vacua: $\varphi(\infty) = -\varphi(-\infty) = \gamma$. For $m = 0$, charge conjugation symmetry is preserved, so that positive and negative energy states are paired, but in addition to continuum modes there is now also a zero-energy bound state localized at the soliton jump. This state is charge self-conjugate and results in a doubly degenerate soliton sector vacuum. The number operator in the presence of the soliton reads [4]

$$N = a^\dagger a - \frac{1}{2} + \int dk\,(b_k^\dagger b_k - c_k^\dagger c_k), \qquad (3)$$

where b_k and c_k denote annihilation operators for continuum fermion and antifermion modes, respectively, while the operators a and $a^\dagger$ couple the two degenerate zero-energy ground states. The ground-state soliton states possess fractional fermion numbers $\pm 1/2$. The Hamiltonian is diagonal in the number representation and *all* the fermion eigenstates display half integral eigenvalues with vanishing fluctuations. The fractional part of the fermion number has a topological character: it is insensitive to local deformations of the bosonic field, depending only on its asymptotic behavior. For $m \neq 0$, charge conjugation symmetry of the Dirac Hamiltonian (1) is broken and the positive and negative energy states are no longer coupled in a simple way. The particle number of the soliton vacuum is $\langle 0|N|0\rangle = -1/\pi \arctan(\hbar g\gamma/mc^2)$ and may exhibit arbitrary fractional eigenvalues [14].

In our scheme to realize particle number fractionalization in atomic gases we consider neutral FD atoms loaded in a periodic optical lattice. The confining optical potential is induced by means of the ac Stark effect of the off-resonant laser beams [12]. We assume a FD gas with two internal levels $\uparrow$ and $\downarrow$ coupled via an em-induced transition. The coupling could be a far-detuned optical Raman transition via an intermediate atomic level, a microwave, or a rf transition. Furthermore, we assume that the two species experience optical potentials which are shifted relative to each other by $\lambda/4$, where λ is the wavelength of light of the confining optical lattice. This is realized, e.g., when the laser beam is blue detuned from the internal transition of the atoms in level $\uparrow$, and red detuned by the same amount from the internal transition of the atoms in level $\downarrow$. A simple example of a 1D lattice potential in that case is $V_\uparrow(x) = V_0 \sin^2(kx)$, and $V_\downarrow(x) = -V_0 \sin^2(kx)$. The neighboring lattice sites represent atoms in different internal levels and are separated by a distance $\lambda/4$. The Hamiltonian for this two-species FD gas is

$$H/\hbar = \frac{\delta}{2}\sum_i(\alpha_i^\dagger\alpha_i - \beta_i^\dagger\beta_i) - \sum_{k\,\mathrm{odd}}(\kappa\alpha_k^\dagger\beta_{k+1} + \mathrm{H.c.}) - \sum_{l\,\mathrm{even}}(\kappa\alpha_{l+1}^\dagger\beta_l + \mathrm{H.c.}). \qquad (4)$$

Here α_i and β_j denote annihilation operators for atoms in levels $\uparrow$ and $\downarrow$, at lattice sites i and j, respectively. The em-induced coupling between the two internal states is described by $\kappa = \int d^3r\, \psi_\uparrow^*(\mathbf{r} - \mathbf{r}_k)\Omega(\mathbf{r})\psi_\downarrow(\mathbf{r} - \mathbf{r}_{k\pm 1})$, and δ stands for the effective detuning between the levels. The em-coupled terms are the analogs of the hopping terms in the corresponding polymer Hamiltonian [4]. The mode functions of the individual lattice sites (Wannier functions) are denoted by $\psi_j(\mathbf{r} - \mathbf{r}_i)$. We assume that the em coupling between the internal levels with frequency $\Omega(\mathbf{r})$ is the only transition mechanism for the atoms between neighboring lattice sites and therefore we ignored the direct tunneling. For simplicity, we also ignore the s-wave scattering between the two FD species.

To produce fermion number fractionalization, we propose to take the coupling frequency $\Omega(\mathbf{r})$ to be a phase-coherent superposition of a standing em field along the x axis and a field with the spatial profile $\varphi(x)$:

$$\Omega(\mathbf{r}) = i\mathcal{V}(\mathbf{r})[\sin(2kx) + \varphi(x)], \qquad (5)$$

with $k \equiv 2\pi/\lambda$ [15]. We show that such a coupling frequency converts the atomic lattice Hamiltonian (4) into the Dirac Hamiltonian (1) in the continuum limit. This is similar to the 1D polymer case [4,7], but the physics is very different: in our atomic system the kink $\varphi(x)$ appears in the em-induced coupling between the internal atomic states, and not as a physical domain wall kink.

For simplicity, we assume that $\eta(x) \equiv \int d^3r\, \psi_\uparrow^*(\mathbf{r} - \mathbf{r}_k)\mathcal{V}(r)\psi_\downarrow(\mathbf{r} - \mathbf{r}_{k\pm 1})$ does not change its sign over the length of the lattice. The purpose of the standing wave in Eq. (5) is to introduce an alternating sign between the neighboring lattice sites. Such a coupling may be prepared by a two-photon optical Raman transition (Fig. 1). The strength of an off-resonant two-photon Rabi frequency in the limit of large detuning, Δ, from the intermediate state is $\Omega \propto \mathcal{R}_1\mathcal{R}_2/\Delta$, where $\mathcal{R}_i$ denote the Rabi frequencies in the individual transitions [16]. For two standing-wave one-photon couplings displaced from one another by $\lambda/4$, the two-photon Rabi frequency is $\Omega \propto \sin(kx)\cos(kx) \propto \sin(2kx)$.

We choose the field profile $\varphi(x)$ to exhibit a phase jump of π at $x = 0$. The phase jump represents a topological phase singularity, or a phase kink. This type of coupling might be produced either by making use of the (rf or microwave) transition between the spin states or by means of an optical Raman transition other than the one used to produce the $\sin(2kx)$ standing wave. A phase profile with topological properties similar to $\varphi(x)$ could be prepared, e.g., by means of a standing microwave $\propto \sin(qx)$ with $q \ll k$. It should also be possible to shape the wave fronts of the coupling lasers to produce a two-photon transition with a desired phase jump. To avoid rapid phase variation at the length scale λ, one could use either laser beams copropagating along the x axis or beams with the wave vectors nearly perpendicular to x. Alternatively, the coupling could possibly also be obtained by a dc magnetic field $B(x) \propto x$ as explained in Ref. [17]. An em field, topologically similar to $\varphi(x)$, was also used to create solitons in atomic condensates [9]. Unlike in Ref. [9] we could assume that the coupling field *itself* is formed by two rapidly rotating em fields resulting in a desired time-averaged phase profile.

We assume that $\sin(2kx)$ and $\varphi(x)$ in Eq. (5) are approximately constant over the spatial overlap area of neighboring lattice site atom wave functions. Then the sine function is approximately ± 1 at each overlap area:

$$\kappa \simeq i\eta[(-1)^n + \varphi(x)]. \qquad (6)$$

For notational simplicity, we take $\kappa^* = -\kappa$, and η real.

In this paper we study the Hamiltonian (4) [with $\Omega(\mathbf{r})$ defined in Eqs. (5) and (6)] only in the continuum limit, where it can be transformed into the relativistic Dirac Hamiltonian (1) exhibiting fractional charge [3]. It may be shown that the fractional eigenvalues also emerge in a more general case, but the continuum field theory is amenable to a simpler description. The continuum limit corresponds to the linearization of the fermionic band structure and becomes accurate in the dilute gas limit, where the atomic correlation length is much larger than the lattice spacing. In the continuum limit we write the fermionic annihilation operators as continuous functions of the lattice spacing $d \equiv \lambda/4$:

$$\alpha_j \equiv \sqrt{2d}\, u(jd), \qquad \beta_j \equiv \sqrt{2d}\, v(jd). \qquad (7)$$

Then the continuum limit proceeds exactly as in the polymer case [4]. To leading order in small d, we obtain

$$H/\hbar = 2id^2\eta \sum_n [u^\dagger(nd)v'(nd) + v^\dagger(nd)u'(nd)] + \frac{\delta d}{2} \sum_n [u^\dagger(nd)u(nd) - v^\dagger(nd)v(nd)] + 2i\eta d \sum_n \varphi(nd)[u^\dagger(nd)v(nd) + v^\dagger(nd)u(nd)]. \qquad (8)$$

Here $u'(nd)$ denotes a discrete spatial derivative of u. In the continuum limit we replace $nd \rightarrow x$ and $d\sum_n \rightarrow \int dx$. By introducing the spinor $\Psi(x) \equiv [u(x)v(x)]^T$ and the transformation, $\Psi \rightarrow \exp(i\pi\sigma^3/4)\Psi$, we may express Eq. (8), with $H = \int dx\, \mathcal{H}$, as the relativistic Dirac Hamiltonian (1), when we identify $c = \lambda\eta/2$, $g = 2\eta$, and $m = 2\hbar\delta/(\lambda^2\eta^2)$. In this case the spinor components refer to the two internal atomic levels. Note that for nonzero detuning δ the system is not charge conjugation symmetric, and the eigenvalues can have any fractional value. The ratio between the coupling strength $\gamma\eta$ and δ therefore determines the fractional part of the particle number. In experiments this could be engineered accurately, allowing a controlled way of preparing the fractional part of the eigenvalues.

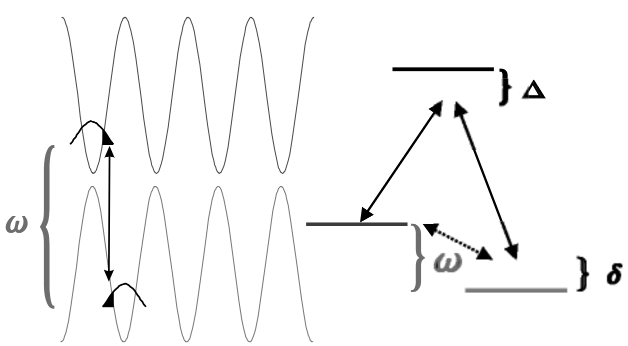

FIG. 1. The energy diagram of the two-species Fermi-Dirac gas in an optical lattice. The atoms occupy two different internal levels and experience different periodic optical potentials shifted by $\lambda/4$ (left). The coupling between the neighboring lattice sites, representing different spin components, is induced by two-photon Raman transitions, or by a superposition of the Raman transition and a one-photon microwave or rf transition (dotted line). The transition region is denoted by the dark shaded areas representing the overlap region of the atomic wave functions. The energy difference between the two components in the overlap region is denoted by ω. The two-photon transition is far detuned by Δ from an intermediate atomic level (right).

The crucial part of our proposal for the fractional particle number is the em field $\varphi(x)$ in Eq. (5). This is very different from the fermion particle number fractionalization in polymers, as our fermionic fields are not coupled to a bosonic matter field with a domain wall soliton. Instead, the coherent em field with a topologically appropriate phase profile is coupled to the FD atoms via internal transitions. This results in the quantization of the FD atomic gas with nontrivial topological quantum numbers corresponding to the soliton sector of the relativistic $1 + 1$ quantum field theory models of fractionalization. On the other hand, a spatially constant phase profile $\varphi(x)$ represents the FD vacuum sector exhibiting integer particle numbers and no bound state.

The normalizable bound state, which plays an important role in the fractionalization, belongs to only one of the two fermionic components, independently of the shape or the position of the phase kink: for the solitonlike phase kink, $\varphi(h) = -\varphi(-h) = \gamma$, for all $h \gg d$, only level $\uparrow$ is occupied, and, for the antisolitonlike phase kink, $\varphi(h) = -\varphi(-h) = -\gamma$, only level $\downarrow$ is occupied [4,7]. When $m = 0$, the bound state has zero energy. Its spatial profile $\sim \exp(-\gamma|x|/d)$ (for a sharp kink at $x = 0$) depends on the relative strength of the superposed em fields, according to Eq. (5), determining γ. Unlike in the polymer case, where the size of the bound state is fixed, in the atomic case this could be varied experimentally.

Because the local density of states is conserved, a zero-energy mode creates a fractional deficit of states in both the valence and the conduction bands. In the presence of charge conjugation symmetry, the density of states is a symmetric function of the energy, and both bands have a decifit of one-half a state. By assigning the atoms in the conduction and valence bands as "particles" and "antiparticles," respectively, we can interpret the fractional particle number operator as the occupation number difference between the bands. However, because the zero mode always occupies only one spin component at a time, the spin is also fractionized. As an example, the two species may correspond to the eigenstates of a single-particle spin operator along the z axis, σ_z, with eigenvalues ± 1. Then the eigenvalues of the many-particle operator $S_z \equiv \sum_n \sigma_z^{(n)}$, localized around the soliton, are fractional with vanishing fluctuations.

The total number of atoms, of course, must remain an integer and any realistic optical lattice has a finite size. For every fractional particle number located at the phase kink (forming a soliton), some fractional charge is distributed at the boundary of the atomic cloud, or is associated with an antikink. Although the fluctuations of the fractional eigenvalues do not then exactly vanish in a finite lattice, it can be shown that the fluctuations decay exponentially as a function of the size of the system and can be considered negligible, if the size is much larger than the atomic correlation length [7,18]. Therefore, *every* localized measurement of the particle number around the phase kink can yield a fractional result.

In experiments on the fractional fermion number we may detect the bound state or measure a fractional expectation value and determine its fluctuations to ascertain that they are compatible with fractionalization. The FD gas exhibits a gap $\sim 2\eta$ in the excitation spectrum. The phase kink creates a bound mode at the center of the gap, hence excitations at the half gap energy [19]. These midgap transitions [7] could be probed in resonance spectroscopy. The bound state also alters the dynamical structure factor, which may be observed via light scattering [16]. The optical signal may be magnified by simultaneously preparing many phase kinks. The fractional particle number could be detected by measuring the occupation numbers of the individual lattice sites. For instance, a magnetic field gradient may be introduced that causes a detectable change in the spin flip frequency from site to site [20]. Finally, an off-resonance optical probe couples to atom density [16]. Fluctuations in the scattered light should therefore convey information about not only the particle number, but also about its fluctuations.

We can also generalize the proposed scheme to particle number fractionalization in higher dimensional models in 2D or 3D optical lattices with atoms coupled to em fields exhibiting phase profiles similar to topological defects or textures [13]. In relativistic $(2 + 1)$D quantum field theory, a fermionic field coupled to a bosonic field exhibiting a vortex profile results in fermion particle number fractionalization [4]. In our approach this would correspond to a 2D optical lattice of FD atoms coupled by an em field with a nonvanishing phase winding around any closed loop circulating the axis of a vanishing field amplitude. This is also similar to the Raman field used in the JILA experiments to couple two atomic condensates in order to create vortices [10]. We may find an analogy to relativistic $(3 + 1)$D quantum field theory in the presence of a nonvanishing SU(2) topological charge of the bosonic field by means of the em field configurations proposed in Ref. [13] to engineer Skyrmion textures in Bose-Einstein condensates. Analogous techniques could possibly also be used to couple FD atoms to em fields with the structure of 3D monopole defects [21].

We have considered the em background field as a coherent classical field. This is consistent with the adiabatic approximation in the fermionic particle number fractionalization, in which case the quantum fluctuations of the bosonic soliton are ignored [4]. For a two-photon optical Raman coupling between the two species, the decoherence rate per atom is determined by the Rayleigh scattering rate $\sim\Gamma \max(|\mathcal{R}_1|^2, |\mathcal{R}_2|^2)/\Delta^2$, where Γ denotes the natural linewidth. This can be reduced by increasing the detuning Δ, provided that at the same time a sufficient laser intensity for the required tunneling rate $\Omega \sim \mathcal{R}_1\mathcal{R}_2/\Delta$ is available. In the case of a rf coupling the quantum effects of the em field can be safely ignored on the time scale of the experiments.

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