PHYSICAL REVIEW
LETTERS

VOLUME 76 | 8 JANUARY 1996 | NUMBER 2

Quantum Phase of a Bose-Einstein Condensate with an Arbitrary Number of Atoms

Juha Javanainen[1] and Sung Mi Yoo[2]
[1]*Department of Physics, University of Connecticut, Storrs, Connecticut 06269-3046*
[2]*Department of Physics, Kangwon National University, Chunchon, Kangwon-do 200-701, South Korea*
(Received 18 August 1995)

We study the interference of two Bose-Einstein condensates within an elementary model. The detection of the atoms is modeled by adapting the standard theory of photon detection. Even though the condensates are taken to be in number states with no phases whatsoever, our stochastic simulations of atom detection produce interference patterns as would also be predicted on the basis of the phases of the macroscopic wave functions describing the condensates. In statistical mechanics terms, we have devised a method to analyze spontaneous symmetry breaking for an arbitrary (not necessarily larger) number of particles.

PACS numbers: 03.75.Fi, 05.30.–d, 32.80.Pj

The confluence of laser cooling and evaporative cooling [1] has recently lead to the first observations [2] of a weakly interacting Bose-Einstein (BE) condensate. Some of the current theoretical work on the optical properties of the condensate [3] and on the consequences of the interparticle interactions [4,5] will undoubtedly soon be tested experimentally. The analogy to lasers [6] should also guarantee that the phase, coherence, and potential for interference of a BE condensate will attract much attention.

In fact, it is customary to attribute to the condensate a macroscopic wave function [5,7] with a magnitude *and* phase. Essentially, the same approach lends itself to elementary textbook discussions of the Josephson effect [8]. Recognizing this connection, we some time ago predicted oscillatory exchange of atoms between two trapped BE condensates that depends on the phases of the macroscopic wave functions [9]. More recently, we have discovered that no phase is needed at all: The atoms will oscillate even if the condensates are initially in number states, provided the atom numbers are "large enough" [10]. In this Letter we take the next, final, conceptual step. We study the interference of atoms that results when two BE condensates are dropped on top of each other. The example is different from that of Refs. [9] and [10], because in the present case we may adapt a plausible quantum measurement theory for the positions of the atoms from the well-established theory of photon detection. We simulate stochastically the outcome of an experiment. We find that the atoms display an interference pattern as would be deduced from the phases of the wave functions of the condensates, *even though no phases have ever been assumed.* In effect, we are now able to discuss the consequences of spontaneously broken phase symmetry for an arbitrary atom number.

We take N spinless, noninteracting bosons residing on a unit interval in one dimension. The Heisenberg picture field operator is

$$\hat{\psi}(s,t) = \sum_k e^{i(kx-\omega_k t)} b_k \,, \qquad (1)$$

where the sum runs over wave numbers, b_k is the annihilation operator for the mode k, and ω_k is the mode frequency. The N atoms are divided into two condensates, $N/2$ atoms each. We assume that the condensates have been given pushes in opposite directions, so that the one-particle states $\pm\kappa$ have $N/2$ atoms in them. Other one-particle states are empty. We thus write the state vector as

$$|\phi^0\rangle = |(N/2)_{+\kappa}, (N/2)_{-\kappa}\rangle \,. \qquad (2)$$

To simplify the notation further, we arbitrarily set $\kappa = \pi$. Then all of our results are periodic in position with the period of 1. We also take the characteristic frequencies $\omega_{\pm\kappa}$

to be the same, which will remove all time dependence from the results.

We now need a quantum measurement theory for the positions of the atoms. The well-known theory of photon detection [11] furnishes us with a model. In the standard version it is assumed that each photon is absorbed (removed) upon detection, and that the matrix element for photon absorption is independent of photon energy. The theory then produces the joint counting rate at times $t_1, \ldots, t_m$ for photon counters positioned at $\mathbf{r}_1, \ldots, \mathbf{r}_m$ as an m-time correlation function of the electric field operator. *Mutatis mutandis,* we posit that in our case, under the same assumptions, the joint counting rate for m atom detectors is

$$R^m(x_1, t_1; \ldots, x_m, t_m) = K^m \langle \hat{\psi}^\dagger(x_1, t_1) \cdots \hat{\psi}^\dagger(x_m, t_m) \times \hat{\psi}(x_m, t_m) \cdots \hat{\psi}(x_1, t_1) \rangle , \quad (3)$$

a Heisenberg picture expectation value of a product of $2m$ boson field operators. K^m is a constant that embodies the sensitivity of the detectors. The advantages of this form include the fact that $R^m \equiv 0$ for $m > N$; N atoms that are each removed upon detection obviously should not trigger more than N detectors.

Let us assume that all atoms do get recorded. The joint probability density for detecting m atoms at positions $x_1, \ldots, x_m$, $p^m(x_1, \ldots, x_m)$, should then be proportional to the joint counting rate $R^m(x_1, \ldots, x_m)$ from Eq. (3). The constant of proportionality is simply chosen in such a way that the integral of p^m over all position variables is unity, as is appropriate for a probability density. For our quantum model with (1), (2), and (3), the analysis of probability densities boils down to an exercise in combinatories. The joint probabilities are

$$p^m(x_1, \ldots, x_m) = \frac{(N-m)!}{N!} \langle \hat{\psi}^\dagger(x_1) \cdots \hat{\psi}^\dagger(x_m) \hat{\psi}(x_m) \cdots \hat{\psi}(x_1) \rangle \quad (4a)$$

$$= \sum_{q=0}^{[m/2]} \frac{[(N/2)!]^2}{[(N/2-q)!]^2} \frac{(N-2q)!}{N!} C_q^m(x_1, \ldots, x_m) . \quad (4b)$$

Here we define $[m/2] = m/2$ for even m and $[m/2] = (m-1)/2$ for odd m. The functions C_q^m are

$$C_q^m(x_1, \ldots, x_m) = \sum \cos[2\pi(x_{\alpha_1} + \cdots + x_{\alpha_q} - x_{\alpha_{q+1}} - \cdots - x_{\alpha_{2q}})] , \quad (5)$$

where the sum runs over all sets of distinct indices $\{\alpha_1, \ldots, \alpha_{2q}\}$ chosen from the set $\{1, \ldots, m\}$, but taking only one permutation of each q-tuple $\{\alpha_1, \ldots, \alpha_q\}$ and $\{\alpha_{q+1}, \ldots, \alpha_{2q}\}$; we set $C_0^m \equiv 1$.

By construction, the joint probabilities are non-negative and normalized. An explicit calculation shows that they are also compatible:

$$\int p^m(x_1, \ldots, x_{m-1}, x_m) dx_m = p^{m-1}(x_1, \ldots, x_{m-1}) . \quad (6)$$

This condition, which is usually not discussed in the theory of photon detection, is crucial in order that the conventional theory of probability may be relied on. Finally, let us consider the probability p^m as a function of a particular individual variable $x = x_i$ with the other variables held fixed. It is obvious from Eqs. (4b) and (5) that p^m is a linear combination of a constant, $\cos(2\pi x)$, and $\sin(2\pi x)$. Because the probabilities are non-negative, $p^m(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m)$ must thus be a constant multiple of a function of the form

$$p(x) = 1 + \beta \cos(2\pi x + \varphi) . \quad (7)$$

In this case β and φ are parameters that depend on the fixed coordinates x_j with $j \neq 1$.

Our plan is to simulate an experiment by generating an N-tuple of random numbers $x_1, \ldots, x_N$ with the probability distribution $p^N(x_1, \ldots, x_N)$. In general, production of random deviates with a prescribed probability density in N-dimensional space rapidly becomes a hopeless proposition as N increases. The present task, though, is facilitated by the observation that the conditional probability density for x_m with $x_1, \ldots, x_{m-1}$ fixed, $p(x_m | x_1, \ldots, x_{m-1}) = p^m(x_1, \ldots, x_m)/p^{m-1}(x_1, \ldots, x_{m-1})$, is also of the form (7). First, we have $p^1(x) \equiv 1$, so we obtain x_1 as a uniformly distributed random number in the interval $[0, 1]$. Next, having already generated $m-1$ coordinates $x_1, \ldots, x_{m-1}$, we simply calculate $p(x | x_1, \ldots, x_{m-1})$ for two different x, determine the parameters β and φ of the function $p(x)$ in Eq. (7) from the results, and use the ensuing $p(x)$ as the distribution from which to draw the subsequent position x_m. As a technical detail, it is probably unwise to use the combinatoric formulas (4b) and (5) for numerical purposes. Instead, we obtain the probabilities p^m directly as quantum expectation values, as in Eq. (4a). All told, we have an N^3 algorithm for generating $x_1, \ldots, x_N$.

An example is given in Fig. 1(a) for $N = 1000$ atoms. We sort the positions $x_1, \ldots, x_N$ into $n_b = 30$ bins of equal width $\Delta x = 1/n_b$, and plot the histogram of the numbers of atoms falling in each bin using the centers of the bins as the abscissas. We also plot as a continuous line the histogram derived from the probability distribution (7) that gives the best least-squares fit to the simulation histogram, with β and φ treated as the free

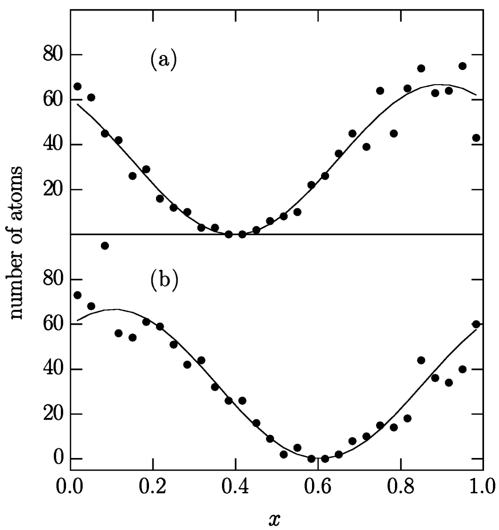

FIG. 1. Numerically simulated histograms (filled circles) for the detected atom positions with $N = 1000$ atoms, for (a) the quantum measurement model and (b) the wave function model. Also shown as solid lines are least-squares fit histograms predicted from the probability distribution of the form $1 + \beta\cos(2\pi x + \varphi)$, with β and φ as the free parameters. In these histograms the positions of the atoms are sorted into $n_b = 30$ equally wide bins.

parameters. Both histograms in effect depict one period of a cosine wave with a nearly 100% modulation depth.

Remarkably, even though the probability density for detecting an *individual* atom $p^1(x) = 1$ has no structure at all, an experiment that records all N atoms at once would nonetheless find an interference pattern with bands of higher and lower atom density. This is a manifestation of the *correlations* between atomic positions embodied in the probabilities p^m. In our example the atom density is essentially of the form $n(x) = n_0[1 + \cos(2\pi x + \varphi)]$. If the experiment were repeated, the result would qualitatively be the same; the phase φ just varies at random from one run to the next.

We now contrast our simulations with the conventional reasoning about the phase of a BE condensate. One would ordinarily grant each condensate a macroscopic wave function, and write the total wave function of the two condensates as

$$\psi(x,t) = \sqrt{\frac{N}{2}}\, e^{-i\omega_\kappa t}(e^{i\pi x + i\phi_+} + e^{-i\pi x + i\phi_-})\,. \quad (8)$$

The phases $\phi_\pm$ are due to spontaneous breaking of phase or "gauge" symmetry [7]. They are independent, fixed for each experiment, but vary randomly from one experiment to the other. In a single experiment with fixed phases $\phi_\pm$, so goes the argument, one expects an atom density of the form $|\psi(x)|^2 = n_0[1 + \cos(2\pi x + \phi_+ - \phi_-)]$; i.e., an interference pattern.

This naive model may be put more rigorously. For instance, one may formally replace the quantum fields describing the condensates by classical fields with the random phases $\phi_\pm$. Alternatively, one may retain the quantum fields, but postulate that the condensates are in the coherent states $|\alpha_\pm\rangle$ with $\alpha_\pm = \sqrt{N/2}\, e^{i\phi_\pm}$ instead of the number states. Whichever way one elects to proceed, conventional arguments lead to the prediction that, as a result of spontaneously broken phase symmetry, the two condensates combine to give an interference pattern with the density $n(x) = n_0[1 + \cos(2\pi x + \phi_+ - \phi_-)]$. We have illustrated this in Fig. 1(b) by plotting the same histograms as in Fig. 1(a) for $N = 1000$ atoms drawn independently from the probability distribution $p(x) = 1 + \cos(2\pi x + \phi_+ - \phi_-)$ for certain fixed values of $\phi_\pm$.

Our measurement theory and the conventional arguments give very similar atom densities [see Figs. 1(a) and 1(b)]. However, there is a crucial conceptual difference. In any derivation based on spontaneous symmetry breaking, the quantity corresponding to the broken symmetry is ultimately *inserted by hand* into the analysis. The phases $\phi_\pm$ are a representative example. On the other hand, the phase φ analogous to $\phi_+ - \phi_-$ emerges as a *result* from our approach. In this sense we have predicted spontaneous symmetry breaking.

Admittedly it is possible to "predict" spontaneous symmetry breaking by assuming the presence of a symmetry breaking field, then going to the thermodynamic limit, and finally letting the symmetry breaking field vanish [7]. A quantity corresponding to the broken symmetry survives this particular sequence of limits without vanishing. However, for a BE condensate the symmetry breaking field is a mathematical fiction and does not correspond to any physical quantity at all. Our earlier approach [10] did away with the symmetry breaking field, but was still based on the limit of large particle number. The novelty of the present work lies in the fact that, by adopting an explicit measurement theory for the positions of the atoms, we have freed our argument from any semblance of the thermodynamic limit as well.

The question to what extent our measurement theoretical predictions and the broken-symmetry predictions can be distinguished in detail elsewhere [12]. Here we offer only a few qualitative remarks. For $N = 1000$ there is no obvious difference between Figs. 1(a) and 1(b). When the number of atoms decreases, the quality of histograms such as those in Fig. 1 deteriorates, and it becomes hard to pick up any interference pattern in the first place. All told, for small N one must fall back on statistical analysis of repeated experiments. The number of repetitions needed to gather enough statistics to distinguish between the two theories increases rapidly with N, and may be expected to be in the thousands for N as small as a few tens.

Our results suggest an intriguing angle to the evolution of the phase of the wave function of a BE condensate: The condensate behaves as if it had a phase as soon as there is a large occupation number of an individual quantum state. No interactions between the atoms are needed to communicate the phase throughout the condensate. Evaporative cooling depends on elastic collisions between the atoms, so this point may seem moot. However, we emphasize that the phase would appear instantaneously even for completely noninteracting atoms if they could be put to the same quantum state with, say, laser cooling. Our views about the role of the interactions are somewhat different from those underlying the ongoing work on the dynamics of BE condensation (see Ref. [13], and references therein).

Our quantum model is clearly simplistic. In recent experiments [2] the condensate was confined to fairly small dimensions, $\sim 1-10\ \mu$m. The condensate is modeled more accurately by a large occupation number of the ground state of an atom trap than of a momentum eigenstate. When released in free space, such a condensate flies apart ballistically. Interference effects are lost on a time scale for which we do not yet have an estimate. Besides, interactions between the atoms, weak as they are, may strongly affect the properties of the condensate [4,5]. Apart from these complications, our thought experiment could, perhaps, be realized by launching two condensates with small momenta toward one another, and letting the combining atom clouds fall on an array of position detectors. Interference is essentially one dimensional, taking place in the direction of the momentum difference between the clouds. Our assumption of one spatial dimension thus has some physical validity, and it could be avoided straightforwardly if a need arises. Finally, the units of length and wave number in our presentation are trivial (and actually somewhat contradictory) conventions. This could be corrected easily, at the expense of some additional notation.

We envisage our ideas leading to general practical tools for the analysis of phase and interference phenomena in BE condensates and atom lasers. For instance, the effects of the finite size of the condensate and of the interactions between the atoms could be studied. A calculation of the entire detection statistics for such situations admittedly seems to be a tall order, but we anticipate that already the lowest correlation functions p^1 and p^2 might give a quantitative estimate of the potential for interference.

In summary, we have presented a new method for the analysis of the interference phenomena associated with a Bose-Einstein condensate. The idea is to compute the joint probability distribution of atom detection for all the atoms at once, and then generate random samples from this distribution for inspection. We have demonstrated that we may predict an interference pattern conventionally attributed to the phase of the condensate without ever assuming a phase. We envisage applications of our ideas to the study of the contrast of the interference, or of the "condensate fraction," also in more complicated situations involving spatial profiles and atom interactions in a condensate. Finally, couched in statistical mechanics language, we have devised a method to investigate spontaneous symmetry breaking for a finite number of particles. There is no need to go to the thermodynamic limit.

This work was triggered by a question asked by W. D. Phillips: Are two light beams in number states able to interfere? Incidentally, a straightforward variant of the argument of the present paper shows that the answer is yes. We acknowledge support from the National Science Foundation.

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