LARGE-ANGLE MOTION OF A SIMPLE PENDULUM

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A bifilar pendulum and a photogate are used to investigate the period of the pendulum as a function of its angular amplitude. The measurements are compared to values evaluated numerically from the equations of motion.

I. INTRODUCTION

The pendulum is perhaps the most studied mechanical system in introductory physics laboratories[1,2]. In most of these experiments; Foucalt's pendulum, Kater's pendulum, physical pendulum and coupled pendula, the amplitude is restricted to small angles so that the period is the familiar result,

$$\tau_0 = 2\pi\sqrt{\frac{L}{g}}, \tag{1}$$

where L is the pendulum length and g is the local acceleration of gravity. In this experiment, we will extend the discussion to include the effects of large angular amplitudes on the motion and period of the pendulum.

II. PERIOD OF THE PENDULUM MOTION

Consider a simple pendulum of mass m suspended by a light, inextensible string of length L as depicted in Fig. 1. The location of the bob is specified by the angular coordinate ϑ.

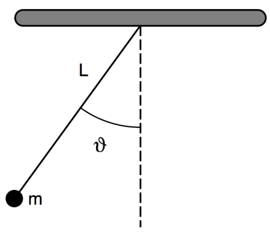

FIG. 1: Schematic diagram of a simple pendulum. The dashed line is the vertical, which is the equilibrium position of the string.

The potential energy of the pendulum in the position shown in Fig. 1 is $U = mgL(1 - cos\vartheta)$

where the zero of potential energy has been chosen to be at the equilibrium position $\vartheta = 0$. The kinetic energy is equal to $\frac{1}{2}m(L\dot{\vartheta})^2$ and hence the total mechanical energy is

$$E = K + U = \frac{1}{2}m(L\dot{\vartheta})^2 + mgL(1 - \cos\vartheta). \tag{2}$$

Consider now the initial condition at $t = 0$ where the bob is released from rest at an initial angle α. The principle of mechanical energy conservation requires that

$$mgL(1 - \cos\alpha) = \frac{1}{2}m(L\dot{\vartheta})^2 + mgL(1 - \cos\vartheta). \tag{3}$$

Rearranging Eq.(3) and taking a square root, the angular velocity of the pendulum is

$$\dot{\vartheta} = \frac{d\vartheta}{dt} = \sqrt{\frac{2g}{L}(\cos\vartheta - \cos\alpha)}. \tag{4}$$

Using the identity, $\cos\alpha = 1 - 2\sin^2(\alpha/2)$, Eq.(4) can be written as

$$\frac{d\vartheta}{dt} = \sqrt{\frac{4g}{L}\left[\sin^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\vartheta}{2}\right)\right]}. \tag{5}$$

This equation can then be integrated from $\vartheta = 0$ to $\vartheta = \alpha$ which is the quarter-period time interval $t = 0$ to $t = \tau/4$. The result is

$$\int_0^{\tau/4} dt = \frac{\tau}{4} = \sqrt{\frac{L}{4g}} \int_0^{\alpha} \left[\sin^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\vartheta}{2}\right)\right]^{-\frac{1}{2}} d\vartheta. \tag{6}$$

There are two features about Eq.(6) which are quite unpleasant. Note first that the integral is improper since the integrand is undefined (infinite) when $\vartheta = \alpha$ at the upper limit. Extreme care must be taken when numerically evaluating such improper integrals. The second point is that it is not obvious that as $\alpha \to 0$, that the period will approach the small angle result τ_0 .

To facilitate the evaluation of the integral of Eq.(6), consider a variable φ such that φ changes from 0 to 2π during one full oscillation of the pendulum. Thus we define φ such that

$$\sin(\varphi) = \frac{\sin(\vartheta/2)}{\sin(\alpha/2}. \tag{7}$$

Show that Eq.(7) can then be used to rewrite Eq.(6) as

$$f(\alpha) = \frac{\tau}{\tau_0} = \frac{2}{\pi} \int_0^{\pi/2} \left(1 - k^2 \sin^2\varphi\right)^{-1/2} d\varphi \tag{8}$$

where $k = sin(\alpha/2)$. This type of integral is called an elliptic integral of the first kind[1]. Use this result to **show that** as the angular amplitude approaches zero, then $\tau \to \tau_0$. Also **show that** for $\alpha \geq 0$, the integrand is positive, thus $\tau \geq \tau_0$. In addition to numerically evaluating Eq.(8), the integrand can be expressed in terms of a series expansion. For $k < 1$, the integrand can be written in terms of a power series and then integrated term by term. Retaining only the first four terms, we have that

$$\frac{\tau}{\tau_0} = 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \ldots . \quad (9)$$

The higher order terms are the amplitude dependent corrections to the approximation of simple harmonic motion. It is clear that they become increasingly important at larger amplitudes. Using the series expansion for $k = \sin(\alpha/2)$, we can write an expansion of $\tau(\alpha)$ in terms of the angular amplitude .

$$\frac{\tau}{\tau_0} = 1 + \frac{1}{16}\alpha^2 + \frac{11}{3072}\alpha^4 + \ldots . \quad (10)$$

III. EXPERIMENTAL METHODS

The bifilar pendulum illustrated in Fig. 2 is used for the measurements instead of a pendulum with a single string. Use the photogate and the computer timing program to measure the

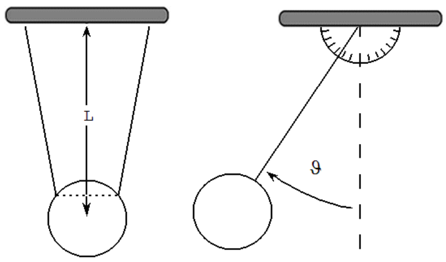

FIG. 2: Front and side views of the bifilar pendulum.

oscillation period of the bifilar pendulum. The values of the initial angle should vary from 5° to 45° in 5° steps. To minimize the uncertainty in the release angle, you should average over four or five measurements. Carefully estimate the small angle (τ_0) period with the smallest value of α you can reliably use.

IV. DATA ANALYSIS

Numerically evaluate the elliptic integral of Eq.(8). Do this for ϑ from 0° to 45° in 5° steps. Check that your answers are correct to at least four places. Tabulate your results along with $\tau/\tau_0 \approx 1 + \alpha^2/16$, which represents the first two terms of the series expansion in Eq.(10) Compare these two numerical results. Plot your measured $\tau(\alpha)$ as a function of α^2. Use the curve to extrapolate to the $\alpha = 0$ intercept and thus determine τ_0.

Equation (8) can be considered as an equation for $\tau(\alpha)$ with τ_0 as the one adjustable parameter. This is a useful approach since τ_0 cannot be measured directly. The variance in τ,

$$\sigma^2 = \frac{1}{N}\sum_i \left(\tau(\alpha_i) - \tau_0 f(\alpha_i)\right)^2, \tag{11}$$

is a minimum with respect to variations in τ_0 for

$$\tau_0 = \frac{\sum \tau(\alpha_i) f(\alpha_i)}{\sum f(\alpha_i)} \tag{12}$$

and that the uncertainty in τ_0 is

$$\Delta = \sigma\sqrt{\frac{1}{\sum f(\alpha_i)^2}}. \tag{13}$$

Compare this value of τ_0 to that obtained previously. Then plot $\tau(\alpha)/\tau_0 - 1$ as a function of α along with your data points.

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[1] Eric W. Weisstein. "Pendulum" From Eric Weisstein's World of Physics–A Wolfram Web Resource. `http://scienceworld.wolfram.com/physics/Pendulum.html`

[2] R.A. Nelson and M.G. Olsson, *The pendulum - rich physics from a simple system*, Am. J. Phys., **54** (1986) ,1-11.

[3] Eric W. Weisstein. "Elliptic Integral of the First Kind." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/EllipticIntegraloftheFirstKind.html`