FOURIER ANALYSIS AND SYNTHESIS

Physics 258/259

I. MATHEMATICAL PRELIMINARIES

In his 1807 essay, *Theory of the Propagation of Heat in Solid Bodies*, the French mathematician J. B. Fourier showed that any piecewise continuous periodic function can be expressed as the sum of an infinite series of sines and cosines whose frequencies are integer multiples of a fundamental frequency ω_0. It was later shown that any function could be expressed as an integral of sines and cosines over all frequencies from 0 to infinity; a relation called the Fourier transform. This remarkable fact has widespread applications in mathematics, physics, and engineering. In this lab we will investigate some of the properties of the Fourier series and Fourier transform and their applications.

Consider first the Fourier series for a periodic function. Let $h(t)$ be a periodic function with period T, that is, $h(t) = h(t+T)$. Fourier's theorem says that $h(t)$ can be expressed as a sum of sine and cosine waves whose frequencies are multiples or harmonics of $f_0 = 1/T$ or $\omega_0 = 2\pi/T$. Mathematically, this series can be expressed as

$$h(t) = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{2\pi n t}{T}\right) + B_n \sin\left(\frac{2\pi n t}{T}\right) \right]. \tag{1}$$

The coefficients A_n and B_n can be found by multiplying both sides of Eq.(1) by either $\cos(2\pi m t/T)$ or $\sin(2\pi m t/T)$ and integrating over a full period T. The results are

$$A_0 = \frac{1}{T} \int_{-T/2}^{T/2} h(t)\, dt, \tag{2}$$

$$A_n = \frac{2}{T} \int_{-T/2}^{T/2} h(t) \cos\left(\frac{2\pi n t}{T}\right) dt, \tag{3}$$

$$B_n = \frac{2}{T} \int_{-T/2}^{T/2} h(t) \sin\left(\frac{2\pi n t}{T}\right) dt. \tag{4}$$

Notice that A_n and B_n have the same units as $h(t)$. Also A_0 in Eq.(2) is equal to the time-averaged value of $h(t)$. If this average is zero, as is frequently the case, then A_0 is zero. Although the integrals shown have limits from $-T/2$ to $T/2$, any limits covering one full period can be used (e.g. α to $\alpha + T$).

This theorem can be simplified if $h(t)$ is a symmetric function so that $h(t) = h(-t)$, or an antisymmetric function for which $h(t) = -h(-t)$. First note that the cosine is a symmetric function while the sine is antisymmetric. If $h(t)$ is a symmetric function, then it will be made up of sums of other symmetric function (cosines) and not symmetric functions (sines). Thus B_n must be zero for all n if $h(t)$ is a symmetric function. Likewise, if $h(t)$ is antisymmetric, then the A_n must be zero for all $n \geq 1$. Many functions can be expressed as either symmetric or anti-symmetric by careful choice of the starting time. Others are inherently asymmetric and must be expressed using both sines and cosines. The Fourier series for some common examples of periodic function are illustrated in the appendix.

The Fourier series of Eq.(1) can also be written in a more compact form with amplitude and phase coefficients C_n and φ_n as

$$h(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{2\pi n t}{T} + \varphi_n\right), \tag{5}$$

where

$$C_0 = A_0,\ C_n = \sqrt{A_n^2 + B_n^2},\ \text{and}\ \varphi_n = \tan^{-1}\left(\frac{-B_n}{A_n}\right) \tag{6}$$

If we plot a graph with the Fourier amplitudes C_n as the ordinate and the frequencies $f_n = n f_0$ as the abscissa, then the resulting graph is the discrete Fourier amplitude spectrum of $h(t)$. Using the Euler identity, $e^{i\theta} = \cos\theta + i\sin\theta$, the Fourier series of Eq.(5) can also be expressed in complex notation, providing an even more compact representation.

The Fourier transform is the extension of the Fourier series to a non-periodic function. The Fourier transform of a function of time in terms of frequency f is given by

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{2\pi i f t} dt. \tag{7}$$

The original function is related to its Fourier transform by the inverse Fourier transform,

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i f t} df. \tag{8}$$

For many examples in experimental physics, signals and spectra are processed in a sampled form as discrete data sets and not as analytic functions, so what we really need is the Discrete Fourier Transform (DFT)[2]. The DFT is simpler mathematically and more relevant computationally than the Fourier transform, although the basic concepts are the same. Specifically, consider a series of N data points h_n taken at times t_n, so that $h_n \equiv h(t_n)$. The time interval between samples is $\delta t = t_{n+1} - t_n$, and thus the sampling frequency is $f_s = 1/\delta t$. The DFT of h(t) is

$$H(f_k) = \sum_{n=0}^{N-1} h(t_n) e^{2\pi i f_k t_n}. \tag{9}$$

The inverse transform for $h(t_n)$ can be written down by inspection,

$$h(t_n) = \sum_{k=0}^{N-1} H(f_k) e^{-2\pi i f_k t_n}. \tag{10}$$

A method to rapidly and efficiently perform the numerical calculation of the DFT, called the Fast Fourier Transform (FFT), was discovered by Cooley and Tukey in 1965[3]. We will use this method to determine the Fourier transform of various waveforms in the lab.

Thus, we now have two mathematically equivalent ways to express any time-dependent amplitude, either as a function of time or as a function of frequency. You will sometimes encounter minor variations of these formulas, since various normalization factors are used by different authors. Variables related by a Fourier transform pair (such as time and frequency or distance and wavevector) are called conjugate variables. In general, $H(f)$ is complex. For voltage signals, the power per unit frequency is proportional to $|H(f)|^2$ and is called the *power spectrum* or *spectral power density* of $h(t)$.

A program written in the C-based LabWindows environment for the National Instruments multifunction DAQ boards, `fft_lw.exe`, will be used to acquire analog data at a sampling rate up to 100 KHz and to display the Fourier spectrum in real time. The program displays the magnitude of the Fourier transform (the square root of the power spectrum).

II. PROCEDURE

A. Square wave synthesis: Use a MathCad worksheet to sum the first N terms of the Fourier series for the square wave illustrated in the Appendix. Set up the worksheet so that it is as general as possible, with expressions for A_n and B_n that can easily be modified. Study graphs showing the sums of harmonics 1+3, 1+3+5, 1+3+5+7, and 1+3+5+7+9, all summed with the proper amplitudes and phases. Use an arbitrary time interval, plotting at least one full period of the square wave. As the approximation to a square wave grows better at higher orders, note the overshoot and ringing that occurs at each of the discontinuities. This is called the Gibbs phenomenon[1].

B. Sawtooth wave synthesis: Derive the Fourier components for a sawtooth wave,

$$h(t) = \frac{2t}{T} \quad \text{for} \quad -\frac{T}{2} < t < \frac{T}{2} \tag{11}$$

and include the derivation in your lab report. Use your worksheet to add up these Fourier components up to at least n=5 and prepare a graph comparing the sum with the exact value of the corresponding sawtooth waveform.

C. Analyzing simple periodic waveforms: To get started with analyzing Fourier spectra, investigate a few simple waveforms using `fft_lw`. Set up a square-wave with a function generator. Adjust the FFT sampling rate to 50,000/s and the number of data sets to either 512 or 1024 (it must always be a power of two). Set the FFT trigger to `SW ANALOG`. Record the Fourier spectrum, using a fundamental frequency f_0 for the square-wave to give an appealing and easily interpreted display, and export it to MathCad. Make a plot (bar-graph) to compare the FFT amplitudes to the calculated Fourier coefficients. Plot only the magnitudes and scale the data such that the first order peak in the FFT is equal in magnitude to the first order Fourier coefficient. How well do they agree? Change the duty cycle from $t_{on}/t_{off} = 1$ for a square-wave to $t_{on}/t_{off} = 2$ for a rectangular wave. Discuss what happens to the harmonic content in the FFT. Repeat for a triangular wave.

D. Tuning Forks: A tuning fork is supposed to provide a (reasonably) pure tone of a specified frequency. Connect a microphone to the input. Analyze the frequency components of two or three tuning forks. How pure are the tones? Does this vary depending on how you strike the fork? Do any of the frequency components damp out more quickly after the tuning fork is struck? Repeat using your own voice. Examine the frequency components as you attempt to sing a single tone. How does the frequency distribution differ for spoken sounds? Make some qualitative comments on the distribution for vowels compared to consonants.

E. Single pulse: Adjust the frequency of a rectangular wave so that its period is about 50 ms. Adjust its duty cycle so that its `ON` state is about 2 ms. The spectrum you observe should have a large peak at 0 followed by little rounded peaks. Compare the minima of this distribution with the zeros of the following formula for the Fourier coefficients for a pulse of amplitude V_0, width τ, repeated at a (slow) period T:

$$|C_n| = \frac{V_0 \tau}{T} \left| \frac{\sin(n\pi\tau/T)}{n\pi\tau/T} \right| \tag{12}$$

Do the peak heights match the predictions as well? What happens as τ goes to zero?

F. Beats and weak signal extraction: Connect two function generators and set both to give a sine wave of approximately equal amplitude of about 0.5 V and a frequency about 3 kHz. Depending on the function generators used, you may need to add series resistors in order to successfully combine the outputs. Describe qualitatively the output as seen on the oscilloscope. Have you seen anything like this before? Change one frequency and observe the effect. Watch the FFT output at the same time and describe what happens. One great advantage of FFT spectral analysis is its ability to isolate a small signal from a large background. Try this by setting one function generator to an amplitude of 1 V and a frequency of 10 kHz (this will be the background), and the other to as small an amplitude as the generator will provide and a frequency of about 5 kHz (this will be the signal). On the oscilloscope, the second signal will be barely detectable, if at all. Now look at the Fourier spectrum. What do you see? Estimate how small a signal could be detected.

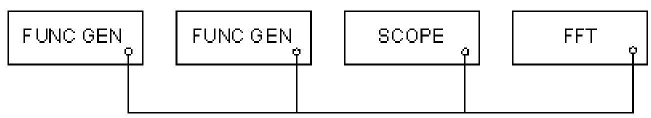

FIG. 1: Two function generators are used to observe beats.

APPENDIX

A1. Square Wave: The square wave illustrate on Fig. A-1 has a duty cycle of $t_{on}/t_{off} = 1$. The wave form is symmetric about t=0, so that one expects that there are no sine terms. By inspection of the figure, the time-average of the wave form is 0, so that $A_0 = 0$. The Fourier series expansion of this wave form, which has only odd harmonics, is

$$h(t) = -\frac{4}{\pi} \sum_{n=1,3,5,...}^{\infty} \frac{(-1)^{\frac{n+1}{2}}}{n} \cos\left(\frac{2\pi n t}{T}\right) . \tag{A-1}$$

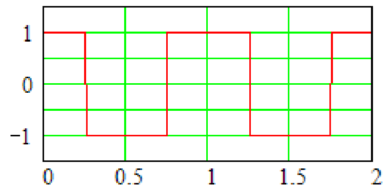

FIG. A-1: A square wave of period $T = 1.0sec$.

A2. Triangle Wave: The triangle-wave illustrate on Fig. A-2 has a duty cycle of $t_{up}/t_{down} = 1$. It is antisymmetric about $t = 0$ and thus contains only sine terms in its Fourier expansion. Also note that the d.c. term (A_0) is zero. The Fourier series for this waveform is

$$h(t) = \frac{8}{\pi^2} \sum_{n=1,3,5,...}^{\infty} \frac{(-1)^{\frac{n+1}{2}}}{n^2} \sin\left(\frac{2\pi n t}{T}\right). \tag{A-2}$$

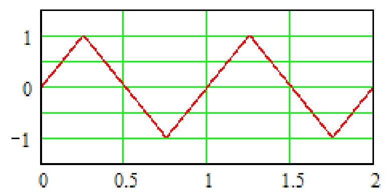

FIG. A-2: A triangle wave.

A3. Sawtooth Wave: The sawtooth wave defined in Eq. 11 is shown in the Fig. A-3 below. It is clearly an antisymmetric function whose time-average value is zero. The derivation of the Fourier series is assigned in part B.

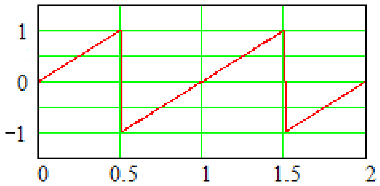

FIG. A-3: A sawtooth wave is a modified triangle wave with an infinite (or zero) duty cycle.

A4. Half-wave rectified sine wave: The half-wave rectified sine wave shown in Fig. A-4 is neither symmetric or antisymmetric and thus we expect both cosine and sine contributions to the Fourier series. Also note that waveform has a non-zero average value. The Fourier series representation is

$$h(t) = \frac{1}{\pi} + \frac{1}{2}\sin\left(\frac{2\pi nt}{T}\right) - \frac{2}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{1}{n^2-1}\cos\left(\frac{2\pi nt}{T}\right). \qquad \text{(A-3)}$$

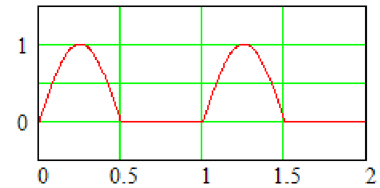

FIG. A-4: The half-wave rectified sine wave.

In comparison, a half-wave rectified cosine wave is symmetric about $t = 0$ and thus contains only cosine terms in addition to the dc term.

Acknowledgments

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[1] W.J. Thompson, *Fourier series and the Gibbs Phenomenon*, Am. J. Phys., **60**, 425-429 (1992).

[2] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, "Fast Fourier Transform." Ch. 12 in *Numerical Recipes in C: The Art of Scientific Computing*, 2nd ed. Cambridge, England: Cambridge University Press, pp. 496-536, 1992.

[3] J. W. Cooley and O. W. Tukey, *An Algorithm for the Machine Calculation of Complex Fourier Series.* Math. Comput. **19**, 297-301, 1965.

[4] G. Arfken, "Fourier Series." Ch. 14 in *Mathematical Methods for Physicists*, 3rd ed. Orlando, FL: Academic Press, pp. 760-793, 1985.

[5] Eric W. Weisstein. "Fourier Series." From *MathWorld*–A Wolfram Web Resource. `http://mathworld.wolfram.com/FourierSeries.html`